Published by Institute of Physics Publishing for SISSA

Received: *September 26, 2006*
Accepted: *October 23, 2006*
Published: *November 17, 2006*

Ramond sector of superconformal algebras via quantum reduction

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Abstract: Quantum hamiltonian reduction of affine superalgebras is studied in the twisted case. The Ramond sector of "minimal" superconformal W-algebras is described in detail, the determinant formula is obtained. Extensive list of examples includes all the simple Lie superalgebras of rank up to 2. The paper generalizes the results of Kac and Wakimoto to the twisted case.

Keywords: Conformal and W Symmetry, BRST Symmetry, Conformal Field Models in String Theory.

*Work partially supported by Minerva Foundation, Germany. Grant No. 8466.

Contents

1. Introduction

Quantum hamiltonian reduction applied to affine superalgebras leads to superconformal W-algebras, which are infinite dimensional algebras with relations that are polynomial in the generators. From the physical point of view the quantum hamiltonian reduction is a procedure of BRST quantization of WZWN models with constraints. Then the W-algebra is a symmetry algebra of the constrained model.

The quantum reduction associates to every $\frac{1}{2}\mathbb{Z}$ gradation on a Lie superalgebra $\mathfrak{g}$ with a non–zero even invariant supersymmetric bilinear form a BRST complex, the homology of which is the W-algebra. Non–equivalent gradations on the same Lie superalgebra lead to different W-algebras. Typical $\frac{1}{2}\mathbb{Z}$ gradations are those generated by $sl(2)$ embeddings into $\mathfrak{g}$: under the action of the $sl(2)$ subalgebra the algebra $\mathfrak{g}$ decouples to a sum of $sl(2)$ eigenspaces with half–integer eigenvalues.

The quantum hamiltonian reduction allows not only to construct W-algebras, but also to describe their representation theory. The characters and the determinant formula of highest weight representations of the underlying affine algebra are translated to the characters and the determinant formula of the correspondent W-algebra representations. Since the theory of Lie superalgebras and their Kac–Moody affinizations is relatively well developed, the quantum reduction becomes a strong tool for study superconformal W-algebras and their representation theory.

The study of hamiltonian reduction of Lie superalgebras has a long history. The classical reduction is known since 1980's [5]. The quantization of the classical reduction is developed in [3, 11, 12, 2]. The first three papers discuss the quantum hamiltonian reduction of $sl(N)$, based on the principal $sl(2)$ embedding to $sl(N)$, which gives rise to the so called W_N algebras [30, 9]. The paper by Bershadsky [2] is on the quantum reduction corresponding to the non-principal $sl(2)$ embedding to $sl(3)$. The reduction results in the so called Bershadsky–Polyakov algebra. In this case the constraints on the WZWN model are of the second class, and "auxiliary fields" ("neutral free superfermions" in the terminology of the present paper) have been introduced to describe the second class constraints.

The quantum reduction procedure was further developed in [16]: the representation theory of the W-algebra was connected to the representation theory of the underlying affine algebra, in particular characters and fusion coefficients of modular invariant representations of W_N algebras were calculated.

The subject of quantum reduction was under intensive study in early 1990's, see for example [10] and references therein. The constrained WZWN models on Lie superalgebras were studied in [15].

The quantum reduction theory was developed for the case of an integral gradation only, or for a half–integral gradation which can be reduced to the integral one. However some Lie superalgebras have only half-integral gradations (including the simplest one $osp(1|2)$). The breakthrough was achieved only in 2003 in the series of papers by Kac et al [21 – 23]: the quantum reduction was constructed for any Lie superalgebra with a non–zero even invariant supersymmetric bilinear form. The structure of the resulting W-algebra was described in detail. W-algebras corresponding to minimal gradations ("minimal" W-algebras) were constructed explicitly. The representation theory of “minimal” W-algebras is developed in [23]: the determinant formula is obtained.

The untwisted case only is discussed in the papers [22, 23]. However the twisted sectors (e.g. the Ramond sector) of superconformal W-algebras are of great importance in physics. In the present paper we generalize the procedure of quantum reduction to the twisted case. The modifications are described in detail. The determinant formula for “minimal” W-algebras is calculated in the twisted case.

The paper is organized as follows. In section 2 we introduce the framework: we recall from [22] the definitions of gradations on Lie superalgebras, “good” gradations, minimal gradations. In section 3 we collect all the necessary information on the main ingredients of the construction: affine vertex algebra, superghost system, neutral free superfermion system. Special attention is devoted to the twisted case. We recall the main points of the general quantum reduction procedure in section 4. The modifications due to twisted case are explained. In section 5 we concentrate on the “minimal” W-algebras. In section 6 we state and prove the determinant formula for the Ramond sector representations of the “minimal” W-algebra. Section 7 contains a list of examples: quantum reduction of Lie superalgebras of rank up to two is briefly discussed and explicit determinant formulas are presented. Section 8 contains the discussion of results and their comparison to the results of [24]. Appendix A fixes the normal ordered product conventions.

When we finished the derivation of the results of the present paper, a work by Kac and Wakimoto [24] appeared on the net. They consider the same subject and obtain essentially the same results as in our paper. However, since there is a conceptual difference in some technical details (see section 8) and in the presentation style, we decided to publish our paper.

The highest root of a Lie superalgebra is conventionally normalized by $(\theta|\theta) = 2$ in the current paper. “$\mathbb{N}$” is used for positive integers, “$\mathbb{N}_0$” – for non-negative integers.

2. Gradation on a Lie superalgebra

We start from a simple finite dimensional Lie superalgebra $\mathfrak{g}$ with a non-degenerate even supersymmetric invariant bilinear form $(.|.)$. The gradation of $\mathfrak{g}$ is the linear space decomposition

$$\mathfrak{g} = \bigoplus_j \mathfrak{g}_j, \qquad \text{such that } [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}. \tag{2.1}$$

We say the gradation is generated by an element $x \in \mathfrak{g}$, if the subspaces $\mathfrak{g}_j$ are eigenspaces of $\operatorname{ad} x$ with eigenvalue j: $[x, u] = j\, u$ for $u \in \mathfrak{g}_j$.

Fix an even element $x \in \mathfrak{g}$, such that it generates a gradation in $\mathfrak{g}$ with half-integer eigenvalues: $\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$. Denote

$$\mathfrak{g}_> = \bigoplus_{j>0} \mathfrak{g}_j, \qquad \mathfrak{g}_\leq = \bigoplus_{j\leq 0} \mathfrak{g}_j. \tag{2.2}$$

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An even element $f \in \mathfrak{g}_{-1}$ is called good if its centralizer $\mathfrak{g}^f = \{u \in \mathfrak{g} \,|\, [f,u]=0\}$ lies in $\mathfrak{g}_{\le}$ ($\mathfrak{g}^f \subset \mathfrak{g}_{\le}$). A gradation is called good if it is generated by an even element $x \in \mathfrak{g}_0$ with half-integer eigenvalues and admits a good element $f \in \mathfrak{g}_{-1}$.

Typical examples of good gradations are gradations associated to the $sl(2)$ embeddings in the Lie superalgebra. They are called Dynkin gradations and generated by an element of an $sl(2)$ triple. Even elements $f, x, e \in \mathfrak{g}$ form an $sl(2)$ triple, if they satisfy the commutation relations:

$$[x,e]=e, \qquad [x,f]=-f, \qquad [e,f]=x. \tag{2.3}$$

It is known from the $sl(2)$ representation theory that the gradation generated by x is a good gradation. There are many good non–Dynkin gradations. Good gradations of simple Lie algebras are classified in [7].

If one chooses $x \in \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, then the root elements of $\mathfrak{g}$ have a well defined grading. The gradation (2.1) generates the root system decomposition: $\Delta = \bigcup_{j \in \frac{1}{2}\mathbb{Z}} \Delta_j$, where

$$\Delta_j = \{\alpha \in \Delta \,|\, \alpha(x) = j\}. \tag{2.4}$$

Define $\Delta_>$ to be a set of roots corresponding to $\mathfrak{g}_>$:

$$\Delta_> = \{\alpha \in \Delta \,|\, \alpha(x) > 0\} = \bigcup_{j>0} \Delta_j\,. \tag{2.5}$$

In this paper we focus on the so called minimal gradations [22]. Minimal gradation is a Dynkin gradation by $\operatorname{ad} x$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \tag{2.6}$$

such that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are even one–dimensional spaces, i.e. $\mathfrak{g}_{-1} = \mathbb{C}f$ and $\mathfrak{g}_1 = \mathbb{C}e$, and $x = [e,f]$.

Minimal gradations are obtained in the Lie algebra case by choosing $sl(2)$ embedding corresponding to the highest root θ: $e = u_\theta, f = u_{-\theta}$, where u_θ is the highest root element of $\mathfrak{g}$. In the Lie superalgebra case the construction is the same, θ is chosen to be the highest root of one of the simple subalgebras of the even part of $\mathfrak{g}$.

Next we want to define the affine vertex algebra $V_k(\mathfrak{g})$ associated to the Lie superalgebra $\mathfrak{g}$. In order to proceed with a BRST quantization we should also introduce two sets of ghost fields: the superghost system and the superfermion system.

3. Ingredients

In this section we introduce the main ingredients of the construction: affine vertex algebra, superghost system, neutral superfermion system. The section may be red and used independently from the other parts of the paper.

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3.1 Affine vertex algebra

Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra with an even nondegenerate supersymmetric invariant bilinear form $(.|.)$. One associates a current $u(z)$ to every $u \in \mathfrak{g}$. The collection of fields $\{u(z)\}_{u\in\mathfrak{g}}$ together with a level $k \in \mathbb{C}$ satisfying the following operator product expansions

$$u(z)\,v(w) = \frac{k\,(u|v)}{(z-w)^2} + \frac{[u,v]\,(w)}{z-w}, \qquad u, v \in \mathfrak{g} \tag{3.1}$$

is called the universal affine vertex algebra $V_k(\mathfrak{g})$.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The Weyl vector ρ is defined with respect to a corresponding set of positive roots:

$$2\,(\rho|\alpha_i) = (\alpha_i|\alpha_i), \qquad i = 1, 2, \ldots, \operatorname{rank}\mathfrak{g}, \tag{3.2}$$

where α_i are simple roots of $\mathfrak{g}$. The Weyl vector can be computed as

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} (-1)^{p(\alpha)}\alpha, \tag{3.3}$$

where the sum is over the set of positive roots Δ_+ and $p(\alpha) = 0$ (respectively 1) for α even (respectively odd).

The dual Coxeter number $h^\vee$ is defined as one half of the eigenvalue of the Casimir operator in the adjoint representation. It can be calculated as

$$h^\vee = (\rho|\theta) + \frac{1}{2}(\theta|\theta), \tag{3.4}$$

where θ is the highest root.

Let $\{u_i\}$ and $\{u^i\}$ be a pair of dual bases of $\mathfrak{g}$, i.e. $(u_i|u^j) = \delta_i^j$. The energy–momentum field for the affine vertex algebra $V_k(\mathfrak{g})$ is given by the Sugawara construction:

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)}\sum_i {:}u^i u_i{:} \tag{3.5}$$

(assuming $k \neq -h^\vee$). The central charge of the Virasoro algebra generated by $L^{\mathfrak{g}}$ is

$$c_{\mathfrak{g}} = \frac{k}{k+h^\vee}\,\mathrm{sdim}\mathfrak{g}. \tag{3.6}$$

The currents $u(z)$ are primary of conformal dimension 1 with respect to $L^{\mathfrak{g}}(z)$. The mode expansion of the affine currents is

$$u(z) = \sum_{n\in\epsilon(u)+\mathbb{Z}} u_n z^{-n-1}, \tag{3.7}$$

where $\epsilon(u) \in \mathbb{R}/\mathbb{Z}$ is called the twisting of the field u. The operator product expansion (3.1) leads to the commutation relations for the modes:

$$[u_m, v_n] = m\,k\,\delta_{m+n,0}\,(u|v) + [u,v]_{m+n}. \tag{3.8}$$

The choice of $\epsilon(u)$ should be consistent with the structure of $\mathfrak{g}$:

$$\epsilon([u,v]) - \epsilon(u) - \epsilon(v) \in \mathbb{Z}. \tag{3.9}$$

In particular $\epsilon(u) = 0, \forall u \in \mathfrak{g}$ (untwisted case) is always allowed. In this paper we will deal only with the case $\epsilon(h) = 0$ for all $h \in \mathfrak{h}$. (Although the case $\epsilon(h) = 1/2$ for some $h \in \mathfrak{h}$ is not forbidden.) In this case all root elements u_α, $\alpha \in \Delta$ have a well defined twisting. Then there is a rank $\mathfrak{g}$ continuous parameter family of twistings, defined as following:

$$\begin{aligned} &\epsilon(h) = 0,\ \ h \in \mathfrak{h}, \\ &\epsilon(u_\alpha) \text{ is any number in } \mathbb{R}/\mathbb{Z},\ \ \alpha \text{ - simple root,} \end{aligned} \tag{3.10}$$

and the twistings for the basis elements corresponding to the non-simple roots are defined by (3.9).

In the untwisted case ($\epsilon(u) = 0,\ \forall u \in \mathfrak{g}$) the modes m and n in the commutation relation (3.8) are integer. Then one recognizes that it is the defining Lie bracket of affine superalgebra $\widehat{\mathfrak{g}}$, the Kac–Moody affinization of $\mathfrak{g}$. The affine superalgebra $\widehat{\mathfrak{g}}$ is defined as an infinite dimensional Lie superalgebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ with commutation relations

$$\begin{aligned} [ut^m, vt^n] &= [u,v]\, t^{m+n} + (u,v)\, m\, \delta_{m+n,0}\, K, \\ [D, at^m] &= mat^m, \qquad [K, \widehat{\mathfrak{g}}] = 0, \end{aligned} \tag{3.11}$$

where $u, v \in \mathfrak{g},\ m, n \in \mathbb{Z}$. Denoting $u_n \equiv ut^n$ and choosing $K = k\,I$ we return to the commutation relation (3.8). D acts on $\widehat{\mathfrak{g}}$ as a minus zero mode of the Sugawara energy–momentum field: $D \sim -L_0^{\mathfrak{g}}$.

The universal affine vertex algebra $V_k(\mathfrak{g})$ written in terms of field modes can be understood as a generalization of the affine algebra $\widehat{\mathfrak{g}}$ to the case of arbitrary twisting. We will denote it by $\widetilde{\mathfrak{g}}$ and call it a *twisted loop algebra*. In many cases the twisted loop algebra $\widetilde{\mathfrak{g}}$ is isomorphic to the untwisted one $\widehat{\mathfrak{g}}$. In the case (3.10) the isomorphism $\widehat{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ is given by $u_{\alpha,n} \mapsto u_{\alpha,n+\epsilon(u)}$, $h(\alpha)_n \mapsto h(\alpha)_n + k\,\epsilon(u_\alpha)\,\delta_{n,0}$, $n \in \mathbb{Z}$, $h(\alpha) \in \mathfrak{h}$ is the Cartan element associated to the root α.

There are different choices of triangular decomposition of $\widetilde{\mathfrak{g}}$. We choose a natural generalization of the triangular decomposition in the untwisted case:[1]

$$\begin{aligned} \widetilde{\mathfrak{g}} &= \widetilde{\mathfrak{n}}^- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}^+, \\ \widetilde{\mathfrak{h}} &= \mathbb{C}[\{h_0 \,|\, h \in \mathfrak{h}, \epsilon(h) = 0\} \cup \{K, D\}], \\ \widetilde{\mathfrak{n}}^+ &= \mathbb{C}[\{u_n \,|\, n > 0, u \in \mathfrak{g}\} \cup \{u_0 \,|\, u \in \mathfrak{n}^+, \epsilon(u) = 0\}], \\ \widetilde{\mathfrak{n}}^- &= \mathbb{C}[\{u_n \,|\, n < 0, u \in \mathfrak{g}\} \cup \{u_0 \,|\, u \in \mathfrak{n}^-, \epsilon(u) = 0\}]. \end{aligned} \tag{3.12}$$

In the case $\epsilon(\mathfrak{h}) = 0$ the set of positive roots $\widehat{\Delta}_+$ of $\widetilde{\mathfrak{g}}$ is a disjoint union of[2]

$$\begin{aligned} &\{(\alpha, 0, m) \,|\, \alpha \in \Delta_-,\ m > 0,\ m \in \epsilon(u_\alpha) + \mathbb{Z}\}, \\ &\{(\alpha, 0, m) \,|\, \alpha \in \Delta_+,\ m \geq 0,\ m \in \epsilon(u_\alpha) + \mathbb{Z}\}, \{(0,0,m) \,|\, m > 0,\ m \in \mathbb{Z}\}, \end{aligned} \tag{3.13}$$

[1] Another choice is implemented in [24]

[2] We denote vectors in the root space by triples $\widehat{\alpha} = (\widehat{\alpha}(\mathfrak{h}), \widehat{\alpha}(K), \widehat{\alpha}(D))$.

where the multiplicity of the last set is $r = \operatorname{rank}\mathfrak{g}$. There are $r+1$ simple roots. The supersymmetric invariant bilinear form of $\mathfrak{g}$ is extended to $\widetilde{\mathfrak{g}}$ in the standard way:

$$\begin{array}{lll} (u_m|v_n) = (u|v)\,\delta_{m+n,0}, & & u,v \in \mathfrak{g}, \\ (D|u_m) = 0, & (K|u_m) = 0, & m \in \epsilon(u) + \mathbb{Z}, \\ (D|D) = 0 = (K|K), & (D|K) = 1, & n \in \epsilon(v) + \mathbb{Z}. \end{array} \tag{3.14}$$

The Weyl vector $\widehat{\rho}$ is defined by the set of $r+1$ equations:

$$2(\widehat{\rho}|\widehat{\alpha}_i) = (\widehat{\alpha}_i|\widehat{\alpha}_i), \qquad \widehat{\alpha}_i - \text{simple roots of } \widetilde{\mathfrak{g}}. \tag{3.15}$$

The Weyl vector $\widehat{\rho}$ is not any more equal to $(\rho, h^\vee, 0)$ in the twisted case.

Conjecture. *Let $\widetilde{\mathfrak{g}}$ be a twisted loop algebra with $\epsilon(\mathfrak{h}) = 0$. Then the Weyl vector $\widehat{\rho}$ defined by* (3.15) *is given by $\widehat{\rho} = (\widetilde{\rho}, h^\vee, 0)$, where*

$$\widetilde{\rho} = \frac{1}{2}\sum_{\alpha\in\Delta_+} (-1)^{p_\alpha}\alpha(1-2\epsilon_\alpha) \tag{3.16}$$

is a "twisted rho", $\epsilon_\alpha \equiv \epsilon(u_\alpha)$.

We will prove this conjecture in a special case only in section 6.4.

A highest weight vector $\big|\,\widehat{\lambda}\,\big\rangle$ of weight $\widehat{\lambda} = (\lambda, k, 0)$ is annihilated by $\widetilde{\mathfrak{n}}^+$ and it is an eigenvector of the generators of the Cartan subalgebra $\widetilde{\mathfrak{h}}$:

$$\begin{array}{ll} \widetilde{\mathfrak{n}}^+\big|\,\widehat{\lambda}\,\big\rangle = 0, & h_0\big|\,\widehat{\lambda}\,\big\rangle = \lambda(h)\big|\,\widehat{\lambda}\,\big\rangle,\ h\in\mathfrak{h}, \\ D\big|\,\widehat{\lambda}\,\big\rangle = 0, & K\big|\,\widehat{\lambda}\,\big\rangle = k\big|\,\widehat{\lambda}\,\big\rangle. \end{array} \tag{3.17}$$

We would like to calculate the eigenvalue of $L_0^{\mathfrak{g}}$ on the highest weight vector $\big|\,\widehat{\lambda}\,\big\rangle$. We do it in the case $\epsilon(\mathfrak{h}) = 0$. In this case $\epsilon(u_\alpha) + \epsilon(u_{-\alpha}) \in \mathbb{Z}$. We will denote $\epsilon_\alpha = \epsilon(u_\alpha)$ (then $\epsilon(u^\alpha) = \epsilon_{-\alpha}$) and choose

$$0 \le \epsilon_\alpha < 1, \text{ for } \alpha\in\Delta_+ \quad \text{and} \quad \epsilon_{-\alpha} = -\epsilon_\alpha. \tag{3.18}$$

In the Cartan–Weyl basis the energy–momentum field is written as

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)}\left(\sum_{i=1}^{r} {:}h^i h_i{:} + \sum_{\alpha\in\Delta} {:}u^\alpha u_\alpha{:}\right). \tag{3.19}$$

Using the formula (A.5) one can express the energy–momentum zero mode as

$$\begin{aligned} L_0^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} &\left(\sum_{i=1}^{r}\left(\sum_{n\in -1-\mathbb{N}_0} h^i_n h_{i,-n} + \sum_{n\in\mathbb{N}_0} h_{i,-n}h^i_n\right)\right. \\ &+ 2\sum_{\alpha\in\Delta_+}\left(\sum_{m\in-\epsilon_\alpha-\mathbb{N}_0} u^\alpha_m u_{\alpha,-m} + (-1)^{p_\alpha}\sum_{m\in 1-\epsilon_\alpha+\mathbb{N}_0} u_{\alpha,-m}u^\alpha_m\right) \\ &\left.+ \sum_{\alpha\in\Delta_+}(-1)^{p_\alpha}\Big(k\,\epsilon_\alpha(1-\epsilon_\alpha) + (1-2\,\epsilon_\alpha)[u_\alpha, u^\alpha]_0\Big)\right). \end{aligned} \tag{3.20}$$

The $L_0^{\mathfrak{g}}$ eigenvalue is

$$L_0^{\mathfrak{g}}|\,\widehat{\lambda}\,\rangle = \frac{1}{2(k+h^\vee)}\Big((\lambda|\lambda+2\widetilde{\rho}) + k\sum_{\alpha\in\Delta_+}(-1)^{p_\alpha}\epsilon_\alpha(1-\epsilon_\alpha)\Big)|\,\widehat{\lambda}\,\rangle\,, \tag{3.21}$$

where ϵ_α, $\alpha\in\Delta_+$ are assumed to be in the range $0\le\epsilon_\alpha<1$ and $\widetilde{\rho}$ is a twisted "rho" defined in (3.16).

Next we would like to generalize the determinant formula to the twisted case. In the untwisted case the determinant formula for the contravariant form on the weight space with weight $\widehat{\lambda}-\widehat{\eta}$ of a Verma module $R_{\widehat{\lambda}}$ with highest weight $\widehat{\lambda}$ is given by (see [20, 23])

$$\det_{\widehat{\eta}}(\widehat{\lambda}) = \prod_{\widehat{\alpha}\in\widehat{\Delta}_+}\prod_{n\in\mathbb{N}}\Big((\widehat{\lambda}+\widehat{\rho}|\widehat{\alpha}) - \frac{n}{2}(\widehat{\alpha}|\widehat{\alpha})\Big)^{q(\widehat{\alpha},n)P(\widehat{\eta}-n\widehat{\alpha})\dim\widehat{\mathfrak{g}}_{\widehat{\alpha}}}\,, \tag{3.22}$$

where $P(\tau)$ is the number of partitions of τ to the sum of positive roots, $\dim\widehat{\mathfrak{g}}_{\widehat{\alpha}}$ is the dimension of the root space $\widehat{\mathfrak{g}}_{\widehat{\alpha}}$ associated to the root $\widehat{\alpha}$, and $q(\widehat{\alpha},n) = (-1)^{p(\widehat{\alpha})(n+1)}$.

The above formula (3.22) is valid also in the twisted case, one has just to use the twisted set of positive roots and the twisted $\widehat{\rho}$, defined by (3.15).

When $\mathfrak{g}$ is a Lie superalgebra there are odd roots which lead to a cancellation of some factors. If γ is an odd isotropic ($(\gamma|\gamma)=0$) root then the correspondent factor does not depend on n, and one can evaluate the product on n explicitly. If β is odd, but not isotropic it is a half of an even root, and then some factors corresponding to β and to 2β cancel each other. As a result one expresses the determinant formula (3.22) in the more explicit way (see [23]):

$$\begin{aligned}\det_{\widehat{\eta}}(\widehat{\lambda}) = (k+h^\vee)^{\sum_{m,n\in\mathbb{N}}P(\widehat{\eta}-(0,0,mn))}\prod_{n\in\mathbb{N}}\prod_{\widehat{\alpha}}\Big((\widehat{\lambda}+\widehat{\rho}|\widehat{\alpha}) - \tfrac{n}{2}(\widehat{\alpha}|\widehat{\alpha})\Big)^{P(\widehat{\eta}-n\widehat{\alpha})}\times\\ \times\prod_{n\in 1+2\mathbb{N}_0}\prod_{\widehat{\beta}}\Big((\widehat{\lambda}+\widehat{\rho}|\widehat{\beta}) - \tfrac{n}{2}(\widehat{\beta}|\widehat{\beta})\Big)^{P(\widehat{\eta}-n\widehat{\beta})}\prod_{\widehat{\gamma}}(\widehat{\lambda}+\widehat{\rho}|\widehat{\gamma})^{P_{\widehat{\gamma}}(\widehat{\eta}-\widehat{\gamma})}\,,\end{aligned} \tag{3.23}$$

where $\widehat{\alpha}=(\alpha,0,m)$ runs on even positive roots, such that $\alpha\ne 0$ and $\frac{1}{2}\widehat{\alpha}$ is not an odd root; $\widehat{\beta}$ runs on odd positive roots, such that $2\widehat{\beta}$ is an even root; $\widehat{\gamma}$ runs on odd positive roots, such that $2\widehat{\gamma}$ is not a root (then $(\widehat{\gamma}|\widehat{\gamma})=0$); $P_{\widehat{\gamma}}$ is a number of partitions not involving $\widehat{\gamma}$.

3.2 Superghost system

Let A be a finite dimensional vector superspace. (In application to the quantum reduction $A=\mathfrak{g}_>$ with flipped parity.) Let $A_{\rm ch} = A\oplus A^*$, define an even skew–supersymmetric non–degenerate bilinear form $<.\,,.>_{\rm ch}$ on $A_{\rm ch}$ by

$$\begin{aligned}<A,A>_{\rm ch} = 0 = <A^*,A^*>_{\rm ch}\\ <a,b^*>_{\rm ch} = -(-1)^{p(a)p(b^*)}<b^*,a>_{\rm ch} = b^*(a)\end{aligned} \tag{3.24}$$

for $a\in A$, $b^*\in A^*$. We introduce a system of local fields $\{c(z),b(z)\}$ ($c\in A, b\in A^*$), called a superghost[3] system, subject to the following operator product expansion:

$$c(z)\,b(w) = \frac{1}{z-w}<c,b>_{\rm ch}. \tag{3.25}$$

[3] "Charged free superfermions" in notation of [22] and [23]; b-c or β-γ system in the physical literature.

The vertex algebra of superghost fields is denoted by $F(A_{\rm ch})$.

Let $\{c_i\}$ and $\{b^i\}$ be the bases of A and A^* such that $<c_i, b^j>_{\rm ch} = \delta_i^j$. Then the superghost system decouples to a set of mutually commuting ghost pairs:

$$c_i(z)\, b^j(w) = \frac{1}{z-w}\, \delta_i^j. \tag{3.26}$$

A family of energy-momentum fields parameterized by $\{\Delta(b^j)\}$ is defined by

$$L^{\rm ch} = -\sum_i \Delta(b^i){:}b^i \partial c_i{:} + \sum_i (1-\Delta(b^i)){:}\partial b^i c_i{:}. \tag{3.27}$$

The field $L^{\rm ch}(z)$ generates the Virasoro algebra with central charge

$$c_{\rm ch} = 2\sum_i (-1)^{p(b_i)} \left(6\Delta(b^i)^2 - 6\Delta(b^i) + 1\right). \tag{3.28}$$

With respect to $L^{\rm ch}$ the ghost field $b^i(z)$ (respectively $c_i(z)$) is primary of conformal dimension $\Delta(b^i)$ (respectively $1-\Delta(b^i)$).

The superghost system is called ϵ–twisted if its fields have the following mode expansions:

$$c_i(z) = \sum_{n\in\epsilon(c_i)+\Delta(b^i)+\mathbb{Z}} c_{i,n} z^{-n-1+\Delta(b^i)}, \qquad b^i(z) = \sum_{n\in\epsilon(b^i)-\Delta(b^i)+\mathbb{Z}} b^i_n z^{-n-\Delta(b^i)}. \tag{3.29}$$

The operator product expansion (3.26) can be written in terms of commutation relations for the modes:

$$\left[c_{i,m}, b^j_n\right] = \delta_i^j \delta_{n+m,0}\,, \qquad \begin{array}{c} m\in\epsilon(c_i)+\Delta(b^i)+\mathbb{Z}, \\ n\in\epsilon(b^j)-\Delta(b^j)+\mathbb{Z}. \end{array} \tag{3.30}$$

We see from here, that $\epsilon(b^i)+\epsilon(c_i)\in\mathbb{Z}$. We will choose $\epsilon(b^i) = -\epsilon(c_i)$.

A vacuum vector $\left|\,0\,\right\rangle_{\rm ch}$ is defined by the set of conditions:

$$\begin{aligned} c_{i,m}\left|\,0\,\right\rangle_{\rm ch} &= 0, \quad m\geq\Delta(b^i), \\ b^i_n\left|\,0\,\right\rangle_{\rm ch} &= 0, \quad n>-\Delta(b^i). \end{aligned} \tag{3.31}$$

The energy-momentum zero mode becomes

$$\begin{aligned} L_0^{\rm ch} = \sum_i \Bigg(&-(-1)^{p(b_i)}\frac{\epsilon(c_i)}{2}(2\Delta(b^i)+\epsilon(c_i)-1) - \sum_{m\in-\Delta(b^i)-\epsilon(c_i)-\mathbb{N}_0} m\, b^i_m c_{i,-m} \\ &-(-1)^{p(b_i)} \sum_{m\in-\Delta(b^i)+1-\epsilon(c_i)+\mathbb{N}_0} m\, c_{i,-m} b^i_m \Bigg). \end{aligned} \tag{3.32}$$

The first term only contributes to the vacuum energy, assuming ϵ is taken in the range $0\leq\epsilon(c_i)<1$ ($\epsilon(c_i)=0$ corresponds to the untwisted case), i.e.

$$L_0^{\rm ch}\left|\,0\,\right\rangle_{\rm ch} = \sum_i \left(-(-1)^{p(b_i)}\frac{\epsilon(c_i)}{2}(2\Delta(b^i)+\epsilon(c_i)-1)\right)\left|\,0\,\right\rangle_{\rm ch}. \tag{3.33}$$

See also ref. [6] where the fermionic and bosonic ghost systems are also discussed in the case of twisted boundary conditions.

3.3 Neutral free superfermion system

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a finite dimensional superspace with a nondegenerate skew-symmetric even bilinear form $<.\,,.>_{\rm ne}$, i.e. it is skew-symmetric on $A_{\bar{0}}$ and symmetric on $A_{\bar{1}}$ and $<A_{\bar{0}}, A_{\bar{1}}>_{\rm ne} = 0$. A set of fields $\{\psi(z)\}_{\psi \in A}$ is called a system of neutral free superfermions, if the fields satisfy the following operator product expansions:

$$\psi(z)\,\phi(w) \sim \frac{1}{z-w} <\psi, \phi>_{\rm ne}, \quad \psi, \phi \in A. \tag{3.34}$$

The vertex algebra of neutral free superfermions is denoted by $F(A_{\rm ne})$. In application to the quantum reduction $A = \mathfrak{g}_{1/2}$ and the bilinear form is defined by

$$<u, v>_{\rm ne} = (f|\,[u, v]), \tag{3.35}$$

where $u, v \in \mathfrak{g}_{1/2}$ and $f \in \mathfrak{g}_{-1}$ is a good element.

The energy–momentum field for the neutral free superfermion system is

$$L^{\rm ne} = \frac{1}{2} \sum_i {:}\partial\psi^i \psi_i{:} = \frac{1}{2} \sum_i (-1)^{p(\psi_i)} {:}\psi_i \partial\psi^i{:}, \tag{3.36}$$

where $\{\psi_i\}$ and $\{\psi^i\}$ are dual bases of A:

$$<\psi_i, \psi^j>_{\rm ne} = \delta_i^j \quad \left(\text{then } <\psi^i, \psi_j>_{\rm ne} = -\delta_i^j (-1)^{p(\psi_i)}\right). \tag{3.37}$$

The central charge of the Virasoro algebra generated by $L^{\rm ne}$ is

$$c_{\rm ne} = -\frac{1}{2}\,\text{sdim} A. \tag{3.38}$$

The neutral free superfermions are primary fields of conformal dimension 1/2 with respect to $L^{\rm ne}$.

The superfermion fields have the following mode expansions:

$$\psi(z) = \sum_{n \in \epsilon(\psi) - 1/2 + \mathbb{Z}} \psi_n z^{-n-1/2}. \tag{3.39}$$

Commutation relations derived from (3.34) read

$$\begin{gathered} [\psi_n, \phi_m] = <\psi, \phi>\, \delta_{n+m,0}, \qquad \psi, \phi \in A, \\ n \in \epsilon(\psi) - 1/2 + \mathbb{Z},\ m \in \epsilon(\phi) - 1/2 + \mathbb{Z}. \end{gathered} \tag{3.40}$$

The consistency condition on twistings is

$$\epsilon(\psi) + \epsilon(\phi) \in \mathbb{Z}, \text{ if } <\psi, \phi> \neq 0. \tag{3.41}$$

The vacuum vector $\big|\,0\,\big\rangle_{\rm ne}$ is defined by the following conditions:[4]

$$\psi_n \big|\,0\,\big\rangle_{\rm ne} = 0, \quad n > 0,\ \psi \in A. \tag{3.42}$$

[4] A different set of annihilation operators is chosen in [24].

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If there are zero modes (it happens when $\epsilon(\psi) = 1/2$ for some $\psi \in A$.) one has to specify their action on the vacuum vector in order to complete the definition.

Next we would like to calculate the $L_0^{\rm ne}$ eigenvalue on the vacuum vector. Using formula (A.5) one gets the following expression for the energy–momentum zero mode:

$$L_0^{\rm ne} = \frac{1}{2}\sum_i \Bigg((-1)^{p(\psi_i)} \sum_{n\in -1/2+\epsilon(\psi_i)-\mathbb{N}_0} (n-\frac{1}{2})\psi_{i,n}\psi^i_{-n} + \sum_{n\in 1/2+\epsilon(\psi_i)+\mathbb{N}_0} (n-\frac{1}{2})\psi^i_{-n}\psi_{i,n} - (-1)^{p(\psi_i)}\frac{1}{2}\epsilon(\psi_i)\big(\epsilon(\psi_i)-1\big)\Bigg). \tag{3.43}$$

If $-1/2 < \epsilon(\psi_i) < 1/2$ then ψ_i contributes $-1/4(-1)^{p(\psi_i)}\epsilon(\psi_i)\big(\epsilon(\psi_i)-1\big)$ to the $L_0^{\rm ne}$ vacuum eigenvalue. The case $\epsilon(\psi_i) = 1/2$ should be treated separately. Since $\sum_i \psi_{i,0}\psi_0^i = -\sum_i(-1)^{p(\psi_i)}\psi_0^i\psi_{i,0}$ the first term in (3.43) contributes $-1/8(-1)^{p(\psi_i)}$ to the eigenvalue, and the overall contribution is $-1/16(-1)^{p(\psi_i)}$. Finally we have

$$L_0^{\rm ne}|\,0\,\rangle_{\rm ne} = \left(\sum_i h_i^{\rm ne}\right)|\,0\,\rangle_{\rm ne}, \tag{3.44}$$

where

$$h_i^{\rm ne} = \begin{cases} -1/4(-1)^{p(\psi_i)}\epsilon(\psi_i)\big(\epsilon(\psi_i)-1\big), & -1/2 < \epsilon(\psi_i) < 1/2, \\ -1/16(-1)^{p(\psi_i)}, & \epsilon(\psi_i) = 1/2. \end{cases} \tag{3.45}$$

In particular the $L_0^{\rm ne}$ vacuum eigenvalue is equal to zero when all the superfermions are untwisted, and equal to $-1/16\,\mathrm{sdim}A$ when $\epsilon(\psi) = 1/2$ for all $\psi \in A$.

4. Quantum reduction

The details of the construction can be found in [22, 23], we reproduce here only main points and results.

4.1 Homology complex

Let $\mathfrak{g}$ be a simple Lie superalgebra with a good gradation on it, generated by an element $x \in \mathfrak{g}$, as described in section 2. Then one introduces three types of vertex algebras: the affine vertex algebra $V_k(\mathfrak{g})$ (section 3.1), the superghost algebra $F(A_{\rm ch})$ (section 3.2), and the superfermion algebra $F(A_{\rm ne})$ (section 3.3).

If the Cartan subalgebra $\mathfrak{h}$ is untwisted (this is always assumed in the current paper), then the root elements have a well defined twisting. If in addition one chooses $x \in \mathfrak{h}$, then the root elements have also a well defined grading by x, and it is convenient to use the Cartan–Weyl basis of $\mathfrak{g}$ in the calculations.

Let Δ_+ be the set of positive roots compatible with the chosen gradation, i.e. $\alpha(x) \geq 0$ if $\alpha \in \Delta_+$. Let Δ_j (respectively $\Delta_>$) be a set of roots corresponding to $\mathfrak{g}_j$ (respectively $\mathfrak{g}_>$) as defined in (2.4) and in (2.5).

The base space for the superghost algebra is the $\mathfrak{g}_>$ space with flipped parity. One introduces a b-c pair for each $\alpha \in \Delta_>$. The parity of the b-c pair is $p(b^\alpha) = p(u_\alpha)+1$, i.e. it is odd, if u_α is even, and even if u_α is odd. The parameter $\Delta(b^\alpha) = 1 - \Delta(c_\alpha)$ is chosen to be equal to the gradation:

$$\Delta(b^\alpha) = j \qquad \text{if } u_\alpha \in \mathfrak{g}_j. \tag{4.1}$$

Then the central charge (3.28) of the superghost Virasoro algebra becomes

$$c_{\text{ch}} = -2 \sum_{j \in \frac{1}{2}\mathbb{N}} \text{sdim}\, \mathfrak{g}_j \left(6j^2 - 6j + 1\right). \tag{4.2}$$

For any basis element u_α, $\alpha \in \Delta_{1/2}$ of $\mathfrak{g}_{1/2}$ one should also add a neutral superfermion with the same parity as u_α. The bilinear form on $\mathfrak{g}_{1/2}$ is given by (3.35).

Now we are ready to introduce an odd field $d(z)$ in the vertex algebra $\mathcal{C}(\mathfrak{g}, x, k) = V_k(\mathfrak{g}) \otimes F(A_{\text{ch}}) \otimes F(A_{\text{ne}})$:

$$d = \sum_{\alpha \in \Delta_>} (-1)^{p_\alpha} u_\alpha b^\alpha - \frac{1}{2} \sum_{\alpha,\beta,\gamma \in \Delta_>} (-1)^{p_\alpha p_\gamma} \mathrm{f}_{\alpha\beta}^{\gamma} c_\gamma b^\alpha b^\beta + \sum_{\alpha \in \Delta_>} (f|u_\alpha) b^\alpha + \sum_{\alpha \in \Delta_{1/2}} b^\alpha \psi_\alpha, \tag{4.3}$$

where $p_\alpha = p(u_\alpha)$ and $\mathrm{f}_{\alpha\beta}^{\gamma}$ are structure constants of $\mathfrak{g}$:

$$[u_\alpha, u_\beta] = \mathrm{f}_{\alpha\beta}^{\gamma} u_\gamma. \tag{4.4}$$

The normal ordering is not necessary since all the fields are commutative in the $d(z)$ monomials.

The key feature of the field $d(z)$ is that the singular part of its operator product expansion with itself vanishes:

$$d(z)d(w) = \text{regular in } (z-w). \tag{4.5}$$

The proof can be found in [22] (Theorem 2.1).

Define an operator d_0 on $\mathcal{C}(\mathfrak{g}, x, k)$ to be the first order pole in the operator product expansion of $d(z)$ with a field from $\mathcal{C}$:

$$d(z)\phi(w) = \ldots + \frac{(d_0\phi)(w)}{z-w} + \ldots \tag{4.6}$$

One can deduce from the associativity condition of operator product expansions that d_0 is an odd derivation of an operator product expansion, i.e.

$$d_0[\phi_1\phi_2]^{(q)} = [(d_0\phi_1)\phi_2]^{(q)} + (-1)^{p(\phi_1)}[\phi_1(d_0\phi_2)]^{(q)}, \tag{4.7}$$

where $[AB]^{(q)}$ is a pole of order q in the operator product expansion of A with B:

$$A(z)B(w) = \sum_{l=-N(A,B)+\mathbb{N}_0} [AB]^{(-l)}(z-w)^l. \tag{4.8}$$

In particular d_0 is an odd derivation with respect to the normal ordered product $:\phi_1\phi_2: = [\phi_1\phi_2]^{(0)}$.

The following crucial feature of d_0:

$$d_0^2 = 0 \tag{4.9}$$

is an immediate consequence of (4.5) and (4.7).

Next one builds a homology complex $(\mathcal{C}(\mathfrak{g},x,k),d_0)$ ("BRST cohomology" in physical literature) of vertex algebra $\mathcal{C}$ with respect to d_0. The homology of the complex

$$H(\mathcal{C},d_0)=\mathrm{Ker}\,d_0/\mathrm{Im}\,d_0 \tag{4.10}$$

is a vertex algebra, the quantum reduction of $\mathfrak{g}$ with respect to x. It is denoted $W_k(\mathfrak{g},x)$.

A charge can be assigned to the fields in $\mathcal{C}$:

$$\mathrm{charge}\,V_k(\mathfrak{g})=0,\quad \mathrm{charge}\,F(A_{\mathrm{ne}})=0,\quad \mathrm{charge}\,b=-1,\quad \mathrm{charge}\,c=1. \tag{4.11}$$

Then the vertex algebra $\mathcal{C}(\mathfrak{g},x,k)$ has charge decomposition

$$\mathcal{C}(\mathfrak{g},x,k)=\bigoplus_{m\in\mathbb{Z}}\mathcal{C}_m\,. \tag{4.12}$$

The field $d(z)$ has charge -1, hence d_0 lowers the charge by 1: $d_0(\mathcal{C}_m)\subset\mathcal{C}_{m-1}$ and $(\mathcal{C}(\mathfrak{g},x,k),d_0)$ is a $\mathbb{Z}$-graded homology complex.

4.2 Twist gluing

We have three commuting vertex algebras: $V_k(\mathfrak{g})$, $F(A_{\mathrm{ch}})$ and $F(A_{\mathrm{ne}})$. Each of them can be twisted in a self-consistent way as described in section 3. However in the quantum reduction procedure these twistings should be related. The restrictions come from the demand that the field $d(z)$ should be untwisted. Denote $\epsilon_\alpha=\epsilon(u_\alpha)$, where α is a positive root of $\mathfrak{g}$. Then (since we consider the case, when the Cartan subalgebra is untwisted) we choose $\epsilon(u_{-\alpha})=-\epsilon_\alpha$ for $\alpha\in\Delta_+$. From the first term in $d(z)$ (4.3) we see that $\epsilon(b^\alpha)=-\epsilon_\alpha$ and therefore (see section 3.2) $\epsilon(c_\alpha)=\epsilon_\alpha$. Let $e\in\mathfrak{g}$ be an element dual to f, then we obtain from the third term in (4.3), that the ghost field b associated to e is untwisted and therefore e and f themselves are also untwisted. From the last term one gets that $\epsilon(\psi_\alpha)=\epsilon_\alpha$. Finally, we conclude that all the possible twistings are parameterized by $r=\mathrm{rank}\,\mathfrak{g}$ numbers ϵ_α, α are the simple roots of $\mathfrak{g}$, modulo the condition that $\epsilon(e)=0$.

There are two cases of particular interest in physics: the Neveu–Schwarz (NS) sector and the Ramond sector. These sectors may be defined for a good gradation on any Lie superalgebra $\mathfrak{g}$. The NS sector is simply the untwisted case: $\epsilon(u)=0,\ \forall\,u\in\mathfrak{g}$. The Ramond sector is defined by the following twistings:

$$\epsilon(u)=\begin{cases}0, & u\in\mathfrak{g}_j,\ j\in\mathbb{Z},\\ 1/2, & u\in\mathfrak{g}_j,\ j\in 1/2+\mathbb{Z}.\end{cases} \tag{4.13}$$

4.3 Structure of the W-algebra

Here we reproduce the results of [22, 23] on the structure of $W_k(\mathfrak{g},x)$. The first fact is that the Virasoro algebra is always contained in $W_k(\mathfrak{g},x)$. It is generated by the field $L(z)$:

$$L=L^{\mathfrak{g}}+L^{\mathrm{ch}}+L^{\mathrm{ne}}+\partial x, \tag{4.14}$$

where $L^{\mathfrak{g}},L^{\mathrm{ch}},L^{\mathrm{ne}}$ are the energy–momentum fields from sections 3.1, 3.2, 3.3 respectively. Due to the ∂x term the conformal dimensions of affine currents are shifted from 1 with

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respect to the Virasoro field $L(z)$: the field $u(z)$ is of dimension $1-j$ if $u \in \mathfrak{g}_j$. Then one can easily check that $d(z)$ is of dimension 1 with respect to $L(z)$ and therefore $d_0 L = 0$ and that L is not in $\operatorname{Im} d_0$. The central charge of the Virasoro algebra generated by $L(z)$ is

$$\begin{aligned} c &= c_{\mathfrak{g}} + c_{\mathrm{ch}} + c_{\mathrm{ne}} - 12k(x|x) \\ &= \frac{k\,\mathrm{sdim}\mathfrak{g}}{k+h^\vee} - 2\sum_{j>0} \mathrm{sdim}\,\mathfrak{g}_j \left(6j^2 - 6j + 1\right) - \frac{1}{2}\,\mathrm{sdim}\,\mathfrak{g}_{1/2} - 12k(x|x) \end{aligned} \tag{4.15}$$

The structure of the W-algebra $W_k(\mathfrak{g}, x)$ is described in the main theorem of [23] (Theorem 4.1). Let $\mathfrak{g}^f = \{u \in \mathfrak{g} |\, [u, f] = 0\}$ be the centralizer of f in $\mathfrak{g}$. Denote

$$J^{(v)} = v + \sum_{\alpha,\beta\in\Delta_>} (-1)^{p_\beta} \mathrm{f}_{v\alpha}^{\beta} {:}c_\beta b^\alpha{:}, \tag{4.16}$$

where $v \in \mathfrak{g}$ and $\mathrm{f}_{v\alpha}^{\beta}$ are the structure constants of $\mathfrak{g}$: $[v, u_\alpha] = \sum_{\beta\in\Delta} \mathrm{f}_{v\alpha}^{\beta} u_\beta$. The theorem states that

1. The only nontrivial homology lies in $\mathcal{C}_0$:

$$\begin{aligned} H_l(\mathcal{C}(\mathfrak{g}, x, k), d_0) &= 0, \text{ if } l \neq 0, \\ H_0(\mathcal{C}(\mathfrak{g}, x, k), d_0) &= W_k(\mathfrak{g}, x); \end{aligned} \tag{4.17}$$

2. The W-algebra $W_k(\mathfrak{g}, x)$ is strongly generated by homology classes of fields $J^{\{a_i\}}$ where $a_i \in \mathfrak{g}^f, i = 1, 2, \dots, \dim \mathfrak{g}^f$ is a basis of $\mathfrak{g}^f$ compatible with the gradation;

3. If $a \in \mathfrak{g}_{-j}$ then the field $J^{\{a\}}$ is of dimension $1 + j$ with respect to $L(z)$ and $J^{\{a\}}$ is equal to $J^{(a)}$ plus a linear combination of normal ordered products of the fields $J^{(b)}$, where $b \in \mathfrak{g}_{-s},\ 0 \le s < j$, the fields $\psi_\alpha,\ \alpha \in \Delta_{1/2}$ and their derivatives.

The Virasoro field $L(z)$ (4.14) is in the same homology class as $J^{\{f\}}(z)$, and since $f \in \mathfrak{g}^f$ the field $L(z)$ is always part of the W-algebra $W_k(\mathfrak{g}, x)$.

In the case of a good gradation $\mathfrak{g}^f \subset \mathfrak{g}_\le$, therefore the conformal dimensions of the generating fields are greater or equal to 1. This is in agreement with the result of [8], that dimension 1/2 fields can be factored out from a W-algebra.

4.4 Highest weight modules of the W-algebra

In this section we are going to discuss highest weight representations of W-algebras $W_k(\mathfrak{g}, x)$ in a framework of quantum reduction. The discussion applies to a general twisted case.

A highest weight vector of the vertex algebra $\mathcal{C}(\mathfrak{g}, x, k) = V_k(\mathfrak{g}) \otimes F(A_{\mathrm{ch}}) \otimes F(A_{\mathrm{ne}})$ is given by

$$|\,\lambda\,\rangle_k = |\,\widehat{\lambda}\,\rangle \times |\,0\,\rangle_{\mathrm{ch}} \times |\,0\,\rangle_{\mathrm{ne}}. \tag{4.18}$$

The full highest weight module $Q_k(\lambda)$ is obtained by applying affine, superghost and superfermion creation operators to the highest weight vector.

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One introduces highest weight representations of $W_k(\mathfrak{g},x)$ in the following way. First define the mode expansions of the generating fields:

$$J^{\{u\}}(z) = \sum_{n\in -\Delta(u)+\epsilon(u)+\mathbb{Z}} J^{\{u\}}_n z^{-n-\Delta(u)}, \qquad (4.19)$$

where $\Delta(u) = 1 + j$, if $u \in \mathfrak{g}_{-j}$, is the conformal dimension of the field $J^{\{u\}}(z)$ with respect to the Virasoro field (4.14). The W-algebra highest weight vector is annihilated by positive modes of all the fields forming the W-algebra $(W_n, n > 0, W \in W_k(\mathfrak{g},x))$. One should also treat the zero modes. First choose a set of mutually commuting (in strong or weak sense) zero modes which is called a set of Cartan generators. Two operators are called commutative in weak sense if their commutator is zero modulo terms which annihilate highest weight vectors. (An example of the W-algebra with Cartan generators commutative in weak sense is studied in [28].) The highest weight vector is an eigenvector of the Cartan generators and it is labelled by the eigenvalues of the Cartan operators. Some of the non Cartan zero modes should also annihilate the highest weight vector.

One can check that the positive modes of the fields $J^{\{u\}}(z)$ annihilate the vector $\left|\,\lambda\,\right\rangle_k$ defined in (4.18):

$$J^{\{u\}}_n\left|\,\lambda\,\right\rangle_k = 0, \qquad n > 0,\ u \in \mathfrak{g}^f. \qquad (4.20)$$

So this vector can be chosen as a highest weight vector of the W-algebra.

It is easy to see that the highest weight vector is d_0 closed:

$$d_0\left|\,\lambda\,\right\rangle_k = 0. \qquad (4.21)$$

To get the W-algebra module $M_k(\lambda)$ one should take the d_0 homology of the $\mathcal{C}$ module: $M_k(\lambda) = H(Q_k(\lambda), d_0)$.

The charge decomposition (4.12) is extended to the $Q_k(\lambda)$ module by a field–state correspondence (charge of the highest weight vector $\left|\,\lambda\,\right\rangle_k$ is taken to be zero). Then again only the zero charge homology of the complex $(Q_k(\lambda), d_0)$ is nontrivial (see Theorem 6.2 of [23]), and one has

$$M_k(\lambda) = H_0(Q_k(\lambda), d_0). \qquad (4.22)$$

Suppose there is a singular vector $\left|\,\widehat{s}\,\right\rangle$ in a highest weight module $R_{\widehat{\lambda}}$ of the twisted loop algebra $\widetilde{\mathfrak{g}}$. Then the vector

$$\left|\,s\,\right\rangle_k = \left|\,\widehat{s}\,\right\rangle \times \left|\,0\,\right\rangle_{\rm ch} \times \left|\,0\,\right\rangle_{\rm ne} \qquad (4.23)$$

is a singular vector in the $\mathcal{C}(\mathfrak{g},x,k)$ algebra module $Q_k(\lambda)$. This vector is d_0 closed and is annihilated by positive modes of the $W_k(\mathfrak{g},x)$ algebra generators, therefore (if it is not d_0 exact) the vector $\left|\,s\,\right\rangle_k$ is also a singular vector in the W-algebra module $M_k(\lambda)$.

5. Minimal W-algebras

5.1 Structure

In the case of minimal gradation (see section 2) $\mathfrak{g}^f$ can be easily described:

$$\mathfrak{g}^f = \mathbb{C}f \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0^\natural, \qquad (5.1)$$

where $\mathfrak{g}_0^\natural = \{u \in \mathfrak{g}_0 | (u|x) = 0\}$ is a subspace of $\mathfrak{g}_0$, orthogonal to x with respect to the even invariant bilinear form: $\mathfrak{g}_0 = \mathbb{C}x \oplus \mathfrak{g}_0^\natural$. Then the $W_k(\mathfrak{g}, x)$ algebra is generated by a Virasoro field, a number of dimension 3/2 fields and a number of dimension 1 fields. In the case of minimal gradation $(x|x) = 1/2$, so one can rewrite the formula for the central charge (4.15) in the simple form:

$$c = \frac{k\,\mathrm{sdim}\mathfrak{g}}{k + h^\vee} - 6k + \frac{1}{2}\,\mathrm{sdim}\,\mathfrak{g}_{1/2} - 2. \tag{5.2}$$

The dimension 1 fields are given by (Theorem 2.1 of [23])

$$J^{\{v\}} = J^{(v)} - \frac{1}{2} \sum_{\alpha,\beta \in \Delta_{1/2}} (-1)^{p_\beta} \mathrm{f}_{v\alpha}^{\beta} {:}\psi_\beta \psi^\alpha{:}, \quad v \in \mathfrak{g}_0^\natural\,, \tag{5.3}$$

and the dimension 3/2 fields are given by

$$\begin{aligned} G^{\{v\}} = J^{(v)} - \frac{(-1)^{p_v}}{3} \sum_{\alpha,\beta \in \Delta_{1/2}} {:}\psi^\alpha \psi^\beta \psi_{[u_\beta,[u_\alpha,v]]}{:} + \sum_{\alpha \in \Delta_{1/2}} J^{([v,u_\alpha])} \psi^\alpha \\ - \sum_{\alpha \in \Delta_{1/2}} \big(k(v|u_\alpha) + \mathrm{str}_{\mathfrak{g}_>}(\mathrm{ad}v\,\mathrm{ad}u_\alpha)\big) \partial\psi^\alpha, \quad v \in \mathfrak{g}_{-1/2}\,, \end{aligned} \tag{5.4}$$

where ψ_u means $\sum_\alpha a_\alpha \psi_\alpha$ if $u = \sum_\alpha a_\alpha u_\alpha$.

The explicit form of operator product expansions of the W-algebra, corresponding to the minimal gradation, is given in Theorem 5.1 of [23]. The dimension-1 fields form a subalgebra with operator product expansions:

$$J^{\{a\}}(z) J^{\{b\}}(w) = \frac{(a|b)(k + \frac{1}{2}h^\vee) - \frac{1}{4}\,\mathrm{str}_{\mathfrak{g}_0}(\mathrm{ad}a\,\mathrm{ad}b)}{(z-w)^2} + \frac{J^{\{[a,b]\}}(w)}{z-w}\,, \tag{5.5}$$

where k is the level of $\mathfrak{g}$ and $a, b \in \mathfrak{g}_0^\natural$. If $\mathfrak{g}_0^\natural$ is simple then the subalgebra is an affine vertex algebra in the definition of section 3.1. The operator product expansion of J and G is:

$$J^{\{v\}}(z)\, G^{\{u\}}(w) = \frac{G^{\{[v,u]\}}}{z-w}\,, \tag{5.6}$$

here $v \in \mathfrak{g}_0^\natural$ and $u \in \mathfrak{g}_{-1/2}$. The fusion rule for two dimension 3/2 fields:

$$G \times G = L + J + {:}JJ{:}, \tag{5.7}$$

and their explicit operator product expansion can be found in Theorem 5.1(e) of [23].

The algebras of “minimal” type were studied from a different point of view in [13, 14]. The classification of minimal gradations on simple Lie superalgebras (see tables in Proposition 4.1 of [22]) gives also a classification of minimal W-algebras.

We have the following set of operators generating the W-algebra:

$$\begin{aligned} L_n, &\quad n \in \mathbb{Z}, \\ G_r^{\{u\}}, &\quad u \in \mathfrak{g}_{-1/2},\ r \in 1/2 + \epsilon(u) + \mathbb{Z}, \\ J_m^{\{v\}}, &\quad v \in \mathfrak{g}_0^\natural,\ m \in \epsilon(v) + \mathbb{Z}. \end{aligned} \tag{5.8}$$

The Cartan subalgebra is a span of

$$\{L_0\} \cup \{J_0^{\{v\}} | v \in \mathfrak{h}^\natural\}, \tag{5.9}$$

where $\mathfrak{h}^\natural$ is a subspace of $\mathfrak{h}$, orthogonal to x with respect to the bilinear form $(.|.)$: $\mathfrak{h} = \mathbb{C}x \oplus \mathfrak{h}^\natural$.

The Cartan generators act diagonally on the other generators. One can introduce roots of the W-algebra. They consist of two components: the first is an eigenvalue of $\mathrm{ad}\, J_0^{\{v\}}$, $v \in \mathfrak{h}^\natural$, the second is an eigenvalue of $\mathrm{ad}\, L_0$. Then the root system Δ_W of the minimal W-algebra $W_k(\mathfrak{g}, x)$ is a disjoint union of

$$\begin{gathered} \{(\alpha, m) \,|\, \alpha \in \Delta_0, m \in \epsilon_\alpha + \mathbb{Z}\}, \\ \{(\alpha^\natural, m) \,|\, \alpha \in \Delta_{1/2}, m \in \tfrac{1}{2} + \epsilon_\alpha + \mathbb{Z}\}, \{(0, m) \,|\, m \in \mathbb{Z}\}, \end{gathered} \tag{5.10}$$

where the multiplicity of the last set is $r = \mathrm{rank}\, \mathfrak{g}$, and $\alpha^\natural$ is the orthogonal projection of α: $\alpha^\natural = \alpha - \frac{1}{2}(\alpha|\theta)$.

There exists an anti-involution ω on a minimal W-algebra. It is defined as

$$\begin{aligned} \omega(L_n) &= L_{-n}, & & \\ \omega(J_n^{\{v\}}) &= J_{-n}^{\{v\}}, & & v \in \mathfrak{h}^\natural, \\ \omega(J_n^{\{v_\alpha\}}) &= J_{-n}^{\{v_{-\alpha}\}}, & & \alpha \in \Delta_0, \\ \omega(G_n^{\{u_\beta\}}) &= G_{-n}^{\{u_{-\theta-\beta}\}}, & & \beta \in \Delta_{-1/2}. \end{aligned} \tag{5.11}$$

The proof that it is indeed an anti-involution can be found in section 6 of [23].

5.2 Representation theory

A highest weight vector $\big|\, \Lambda, h \,\big\rangle$ is defined by

$$\begin{aligned} L_n, G_n^{\{u\}}, J_n^{\{v\}} \big|\, \Lambda, h \,\big\rangle &= 0, n > 0, \\ L_0 \big|\, \Lambda, h \,\big\rangle &= h \big|\, \Lambda, h \,\big\rangle \\ J_0^{\{v\}} \big|\, \Lambda, h \,\big\rangle &= \Lambda(v) \big|\, \Lambda, h \,\big\rangle, v \in \mathfrak{h}^\natural. \end{aligned} \tag{5.12}$$

The zero modes which are not in the Cartan subalgebra (if there are such modes) should be treated separately. We want to split Δ_0 and $\Delta_{1/2}$ to positive and negative parts. The splitting of Δ_0 is naturally given by a set of positive roots of $\mathfrak{g}$: $\Delta_0^+ = \Delta_+ \cap \Delta_0$. To split $\Delta_{1/2}$ one has to choose $h_0 \in \mathfrak{h}^\natural$ such that

$$\begin{aligned} &\alpha(h_0) > 0, \quad \forall \alpha \in \Delta_0^+, \\ &\alpha(h_0) \neq 0, \quad \forall \alpha \in \Delta_{1/2} \text{ (except } \alpha = \theta/2). \end{aligned} \tag{5.13}$$

We introduce

$$\begin{aligned} \Delta_j^+ &= \{\alpha \,|\, \alpha \in \Delta_j \text{ and } \alpha(h_0) > 0\}, \\ \Delta_j^- &= \{\alpha \,|\, \alpha \in \Delta_j \text{ and } \alpha(h_0) < 0\}, \end{aligned} \qquad j = -1/2, 0, 1/2. \tag{5.14}$$

The root $\frac{1}{2}\theta$ (if there is such a root) does not belong to $\Delta^+_{1/2} \cup \Delta^-_{1/2}$. Note that there can be a few choices of the $\Delta_{1/2}$ splittings corresponding to the same Δ^+_0.

Now we complete the definition of a highest weight vector:

$$\begin{aligned} J_0^{\{\alpha\}} \big|\, \Lambda, h \,\big\rangle &= 0, \quad \alpha \in \Delta^+_0, \epsilon_\alpha = 0, \\ G_0^{\{\beta\}} \big|\, \Lambda, h \,\big\rangle &= 0, \quad \beta \in \Delta^+_{-1/2}, \epsilon_\beta = 1/2, \end{aligned} \tag{5.15}$$

where we denote $J^{\{\alpha\}} = J^{\{v_\alpha\}}, G^{\{\beta\}} = G^{\{u_\beta\}}$. If $\theta/2 \in \Delta$ and $\epsilon_{\theta/2} = 1/2$, then there is a fermionic operator $G_0^{\{-\theta/2\}}$. It commutes with Cartan generators and therefore the highest weight vector $\big|\, \Lambda, h \,\big\rangle$ is its eigenvector:

$$G_0^{\{-\theta/2\}} \big|\, \Lambda, h \,\big\rangle = g(\Lambda, h) \big|\, \Lambda, h \,\big\rangle. \tag{5.16}$$

The eigenvalue $g(\Lambda, h)$ can be calculated from the $\left[G_0^{\{-\theta/2\}}, G_0^{\{-\theta/2\}}\right]$ bracket.

In the quantum reduction procedure the highest weight vector $\big|\, \Lambda, h \,\big\rangle$ is given by

$$\big|\, \Lambda, h \,\big\rangle = \big|\, \widehat{\lambda} \,\big\rangle \times \big|\, 0 \,\big\rangle_{\rm ch} \times \big|\, 0 \,\big\rangle_{\rm ne}, \tag{5.17}$$

where Λ and h are functions of λ and k, which are calculated below. But first we should complete the definition of the superfermion vacuum $\big|\, 0 \,\big\rangle_{\rm ne}$ (see 3.42) by specifying the action of superfermion zero modes on it in agreement with the second line in (5.15). From the last term in (5.4) and since the dual superfermion is $\psi^\alpha = \psi_{\theta-\alpha}$ we get that

$$\psi_{\alpha,0} \big|\, 0 \,\big\rangle_{\rm ne} = 0, \quad \alpha \in \Delta^+_{1/2},\ \epsilon_\alpha = 1/2. \tag{5.18}$$

It is easy to see that this is also sufficient condition for the second equation in (5.15). The fermion $\psi_{\theta/2}$ is self dual, so $\left[\psi_0^{\theta/2}, \psi_0^{\theta/2}\right] = 1$ (if $u_{\theta/2}$ is appropriately normalized), then $\psi_0^{\theta/2} \big|\, 0 \,\big\rangle_{\rm ne} = \frac{1}{\sqrt{2}} \big|\, 0 \,\big\rangle_{\rm ne}$.

Now we are ready to calculate the $\Lambda(\lambda, k)$ and $h(\lambda, k)$ dependence. From (4.14) we get

$$h = h^{\mathfrak{g}} + h^{\rm ch} + h^{\rm ne} - \frac{1}{2}(\lambda|\theta), \tag{5.19}$$

where (see (3.21), (3.33) and (3.45))

$$h^{\mathfrak{g}} = \frac{1}{2(k + h^\vee)} \Big((\lambda|\lambda + 2\widetilde{\rho}) + \sum_{\alpha \in \Delta_+} (-1)^{p_\alpha} k\, \epsilon_\alpha (1 - \epsilon_\alpha) \Big), \tag{5.20}$$

$$h^{\rm ch} = \frac{1}{2} \sum_{\alpha \in \Delta_{1/2}} (-1)^{p_\alpha} \epsilon_\alpha^2, \tag{5.21}$$

$$h^{\rm ne} = -\frac{1}{4} \sum_{\alpha \in \Delta_{1/2}} (-1)^{p_\alpha} (\epsilon_\alpha - 1)(\epsilon_\alpha - 2\Theta(\epsilon_\alpha - 1/2)). \tag{5.22}$$

All ϵ_α are assumed to be in the range $0 \leq \epsilon_\alpha < 1$; $\Theta(x)$ is the step function:

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases} \tag{5.23}$$

To calculate $\Lambda(v)$ we rewrite (5.3) for the case $v \in \mathfrak{h}^\natural$:

$$J^{\{v\}} = v + \sum_{\alpha\in\Delta_>} (-1)^{p_\alpha}\alpha(v){:}c_\alpha b^\alpha{:} - \frac{1}{2}\sum_{\alpha\in\Delta_{1/2}} (-1)^{p_\alpha}\alpha(v){:}\psi_\alpha\psi^\alpha{:}, \quad v \in \mathfrak{h}^\natural. \tag{5.24}$$

Then using the formula (A.5) and the definitions of $\left|\,0\,\right\rangle_{\rm ch}$ (3.31) and $\left|\,0\,\right\rangle_{\rm ne}$ (3.42)

$$\Lambda(v) = \lambda^\natural(v) - \frac{1}{2}\sum_{\alpha\in\Delta_{1/2}} (-1)^{p_\alpha}\alpha(v)(\epsilon_\alpha + \varkappa_\alpha), \tag{5.25}$$

where again $0 \le \epsilon_\alpha < 1$, and $\varkappa_\alpha$ is the eigenvalue of the $\psi_{\alpha,-1/2+\epsilon_\alpha}\psi^\alpha_{1/2-\epsilon_\alpha}$ operator:

$$\varkappa_\alpha = \begin{cases} 0,\ 0 \le \epsilon_\alpha < 1/2, \\ 0,\ \epsilon_\alpha = 1/2,\ \ \alpha \in \Delta^-_{1/2}, \\ 1,\ \epsilon_\alpha = 1/2,\ \ \alpha \in \Delta^+_{1/2}, \\ 1,\ 1/2 < \epsilon_\alpha < 1. \end{cases} \tag{5.26}$$

and $\lambda^\natural$ is the projection of λ orthogonal to θ:

$$\lambda = \lambda^\natural + \frac{1}{2}(\lambda|\theta)\theta, \quad (\lambda^\natural|\theta) = 0. \tag{5.27}$$

Then $\lambda(v) = \lambda^\natural(v)$, if $v \in \mathfrak{h}^\natural$.

We are interested mainly in two twistings:

$$\text{NS sector:} \qquad \epsilon_\alpha = 0\ \forall\alpha \in \Delta, \tag{5.28}$$

$$\text{Ramond sector:} \qquad \epsilon_\alpha = \begin{cases} 1/2,\ \alpha \in \Delta_{1/2}, \\ 0, \quad\ \alpha \in \Delta_0 \text{ or } \alpha = \theta. \end{cases} \tag{5.29}$$

In the NS sector the modes of dimension-3/2 generators are in $1/2 + \mathbb{Z}$, the modes of other generators are integer. In the Ramond sector all the modes are integer. In these cases the λ, k dependence of weights Λ and h is easily expressed:

$$\text{NS:} \quad \begin{aligned} &\Lambda = \lambda^\natural, \\ &h = \frac{(\lambda|\lambda + 2\rho)}{2(k + h^\vee)} - \frac{1}{2}(\lambda|\theta), \end{aligned} \tag{5.30}$$

$$\text{R:} \quad \begin{aligned} &\Lambda = \lambda^\natural - \rho^\natural_{1/2}, \\ &h = \frac{(\lambda|\lambda + 2\rho_0)}{2(k + h^\vee)} + \operatorname{sdim}\mathfrak{g}_{1/2}\left(\frac{1}{16} + \frac{k}{8(k + h^\vee)}\right) - \frac{(\lambda|\theta)\,(k + h^\vee - 1)}{2(k + h^\vee)}, \end{aligned} \tag{5.31}$$

where $\rho_{1/2}$ and ρ_0 are the "rho" vectors for $\Delta^+_{1/2}$ and Δ^+_0 respectively:

$$\rho_{1/2} = \frac{1}{2}\sum_{\alpha\in\Delta^+_{1/2}} (-1)^{p_\alpha}\alpha, \tag{5.32}$$

$$\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta^+_0} (-1)^{p_\alpha}\alpha = \rho^\natural. \tag{5.33}$$

We introduce a contravariant bilinear form $B(\,|\,a\,\rangle, |\,b\,\rangle)$ (with respect to the anti-involution (5.11)) on the W-algebra Verma module $M_k(\Lambda,h)$ with highest weight vector $|\,\Lambda,h\,\rangle$. Contravariance means that $B(J|\,a\,\rangle, |\,b\,\rangle) = B(|\,a\,\rangle, \omega(J)|\,b\,\rangle)$, for $J \in W_k(\mathfrak{g},x)$ and $|\,a\,\rangle, |\,b\,\rangle \in M_k(\Lambda,h)$. The form is normalized by $B(|\,\Lambda,h\,\rangle, |\,\Lambda,h\,\rangle) = 1$. The kernel of the contravariant form B is a maximal submodule of $M_k(\Lambda,h)$. In the next section we use this property to compute the determinant of this form.

6. Determinant formula for minimal W-algebras

This section contains the main result of the current paper: the determinant formula for minimal W-algebras in the twisted case. First we state the result: the determinant formula for the NS sector (untwisted case) is presented in section 6.1, it was obtained by Kac and Wakimoto in [23] and is also derived in section 6.4. Section 6.2 contains the determinant formula for the Ramond sector, in which modes of all the fields are integer. We also include the lengthy formula for the general twisted case in section 6.3. In section 6.4 we prove these determinant formulas.

Before we proceed to the determinant expressions we would like to remind to the reader some notation: h is an eigenvalue of the Virasoro field zero mode L_0 on the highest weight vector, $\Lambda(v)$ is the eigenvalue of the dimension 1 fields $J_0^{\{v\}}, v \in \mathfrak{h}^\natural$. The number $r = \mathrm{rank}\,\mathfrak{g}$. ρ_0 and $\rho_{1/2}$ are the "rho" vectors for Δ_0^+ and $\Delta_{1/2}^+$ respectively, they are given by (5.32) and (5.33). The partition function $P_W(\widehat{\tau})$ is a number of partitions of $\widehat{\tau}$ to a sum of positive roots of the W-algebra, where as usual odd roots appear maximum one time in the sum. The function $q(\alpha,n)$ in the degrees is

$$q(\alpha,n) = (-1)^{p_\alpha(n+1)}, \tag{6.1}$$

i.e. it is equal to -1 if α is odd and n is even, and equal to 1 otherwise.

We would like to stress here that the determinants are polynomials in Λ and h. If $q(\alpha,n) = -1$ then the degree of the corresponding factor is negative; it means that the factor cancels some other factor as it is explained below. (And exactly as it happens in the twisted loop algebra determinant formula (3.23)). Odd roots of a lie superalgebra are of two types: isotropic ($(\alpha|\alpha) = 0$) or a half of an even root. If the root α is isotropic, then the corresponding factor $\mathcal{N}_{n,m}^{\,\alpha}$ does not depend on n and the product over n can be evaluated explicitly:

$$\prod_{n\in\mathbb{N}} (\mathcal{N}_{n,m}^{\,\alpha})^{q(\alpha,n)P_W(\widehat{\eta}-n(\alpha^\natural,m))} = (\mathcal{N}_{1,m}^{\,\alpha})^{P_W^{\widehat{\alpha}}(\widehat{\eta}-(\alpha^\natural,m))}, \tag{6.2}$$

where $\widehat{\alpha} = (\alpha^\natural, m)$ and $P_W^{\widehat{\alpha}}(\widehat{\tau})$ is the number of partitions of $\widehat{\tau}$ to the sum of the W-algebra positive roots not including the root $\widehat{\alpha}$ itself.

If the root α is a half of an even root, then the factor $\mathcal{N}_{n,m}^{\,\alpha}$ cancels one of the factors corresponding to the root 2α. For example, if there is a root $\theta/2$ then one can express

$$\begin{aligned}\prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,\theta}_{n,m}(k,\Lambda,h)^{P_W(\widehat{\eta}-(0,mn))} \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}+\epsilon_{\theta/2}+\mathbb{N}_0}} \mathcal{N}^{\,\theta/2}_{n,m}(k,\Lambda,h)^{q(\theta/2,n)P_W(\widehat{\eta}-(0,mn))} = \\ = \prod_{\substack{m,n\in\mathbb{N},\\ m-n\in 2\mathbb{Z}+2\epsilon_{\theta/2}}} \nu_{n,m}(k,\Lambda,h)^{P_W(\widehat{\eta}-(0,\frac{mn}{2}))},\end{aligned} \tag{6.3}$$

where $\epsilon_{\theta/2}=0$ in the NS case and $\epsilon_{\theta/2}=1/2$ in the Ramond case, and

$$\nu_{n,m}(k,\Lambda,h) \equiv \mathcal{N}^{\,\theta}_{\frac{n}{2},m}(k,\Lambda,h) = \mathcal{N}^{\,\theta/2}_{n,\frac{m}{2}}(k,\Lambda,h). \tag{6.4}$$

Taking into account all the above remarks we proceed to the determinant formulae.

6.1 NS sector

Let $\widehat{\eta}=(\eta,s), \eta\in\mathfrak{h}^{\natural *}, s\in\frac{1}{2}\mathbb{Z}$. The determinant $\det^{\mathrm{NS}}_{\widehat{\eta}}(k,\Lambda,h)$ of the Verma module with highest weight (Λ,h) on the weight space $(\Lambda-\eta,h+s)$ is given by

$$\begin{aligned}\det{}^{\mathrm{NS}}_{\widehat{\eta}}(k,\Lambda,h) = (k+h^\vee)^{(r-1)\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,\theta}_{n,m}(k,\Lambda,h)^{P_W(\widehat{\eta}-(0,mn))} \\ \times \prod_{\substack{\beta\in\Delta_0\\ m,n\in\mathbb{N}}} \mathcal{N}^{\,\beta}_{n,m}(k,\Lambda)^{q(\beta,n)P_W(\widehat{\eta}-n(\beta,m))} \prod_{\substack{\beta\in\Delta_0^+\\ n\in\mathbb{N}}} \mathcal{N}^{\,\beta}_{n,0}(k,\Lambda)^{q(\beta,n)P_W(\widehat{\eta}-n(\beta,0))} \\ \times \prod_{\substack{\alpha\in\Delta_{1/2}\\ m\in\frac{1}{2}+\mathbb{N}_0, n\in\mathbb{N}}} \mathcal{N}^{\,\alpha}_{n,m}(k,\Lambda,h)^{q(\alpha,n)P_W(\widehat{\eta}-n(\alpha^\natural,m))},\end{aligned} \tag{6.5}$$

where

$$\mathcal{N}^{\,\beta}_{n,m}(k,\Lambda) = (\Lambda+\rho_0|\beta) + m(k+h^\vee) - \tfrac{n}{2}(\beta|\beta), \qquad \beta\in\Delta_0, \tag{6.6}$$

$$\begin{aligned}\mathcal{N}^{\,\alpha}_{n,m}(k,\Lambda,h) = h - \frac{1}{4(k+h^\vee)}\Big(\big(2(\Lambda+\rho_0|\alpha^\natural) + 2m(k+h^\vee) - n(\alpha|\alpha)\big)^2 \\ + 2(\Lambda|\Lambda+2\rho_0) - (k+1)^2\Big), \qquad \alpha\in\Delta_{1/2},\end{aligned} \tag{6.7}$$

$$\mathcal{N}^{\,\theta}_{n,m}(k,\Lambda,h) = h - \frac{1}{4(k+h^\vee)}\Big(\big(m(k+h^\vee)-n\big)^2 + 2(\Lambda|\Lambda+2\rho_0) - (k+1)^2\Big). \tag{6.8}$$

The partition function $P_W(\eta)$ is defined with respect to the root system Δ_W of the untwisted (NS) sector of the vertex algebra $W_k(\mathfrak{g},x)$. The set of positive roots Δ_W^+ is a disjoint union of

$$\begin{gathered}\{(\alpha,m)|\alpha\in\Delta_0, m\in\mathbb{N}\}, \{(\alpha,0)|\alpha\in\Delta_0^+\}, \\ \{(\alpha^\natural,m)|\alpha\in\Delta_{1/2}, m\in\tfrac{1}{2}+\mathbb{N}_0\}, \{(0,m)|m\in\mathbb{N}\},\end{gathered} \tag{6.9}$$

where the multiplicity of the last set is $r=\operatorname{rank}\mathfrak{g}$.

6.2 Ramond sector

Let $\widehat{\eta} = (\eta, s), \eta \in \mathfrak{h}^{\natural *}, s \in \mathbb{Z}$. The determinant $\det^{\mathrm{R}}_{\widehat{\eta}}(k, \Lambda, h)$ of the Verma module with highest weight (Λ, h) on the weight space $(\Lambda - \eta, h + s)$ is given by

$$\begin{aligned}\det^{\mathrm{R}}_{\widehat{\eta}}(k, \Lambda, h) &= (k + h^\vee)^{(r-1)\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,\theta}_{n,m}(k, \Lambda, h)^{P_W(\widehat{\eta}-(0,mn))} \\ &\times \prod_{\substack{\beta\in\Delta_0\\ m,n\in\mathbb{N}}} \mathcal{N}^{\,\beta}_{n,m}(k, \Lambda)^{q(\beta,n)P_W(\widehat{\eta}-n(\beta,m))} \prod_{\substack{\beta\in\Delta_0^+\\ n\in\mathbb{N}}} \mathcal{N}^{\,\beta}_{n,0}(k, \Lambda)^{q(\beta,n)P_W(\widehat{\eta}-n(\beta,m))} \\ &\times \prod_{\substack{\alpha\in\Delta_{1/2}\\ m,n\in\mathbb{N}}} \mathcal{N}^{\,\alpha}_{n,m}(k, \Lambda, h)^{q(\alpha,n)P_W(\widehat{\eta}-n(\alpha^\natural,m))} \prod_{\substack{\alpha\in\Delta_{1/2}^+\\ n\in\mathbb{N}}} \mathcal{N}^{\,\alpha}_{n,0}(k, \Lambda, h)^{q(\alpha,n)P_W(\widehat{\eta}-n(\alpha^\natural,0))},\end{aligned} \tag{6.10}$$

where

$$\mathcal{N}^{\,\beta}_{n,m}(k, \Lambda) = (\lambda^\natural + \rho_0|\beta) + m(k + h^\vee) - \tfrac{n}{2}(\beta|\beta), \qquad \beta \in \Delta_0, \tag{6.11}$$

$$\begin{aligned}\mathcal{N}^{\,\alpha}_{n,m}(k, \Lambda, h) = h &- \frac{1}{4(k+h^\vee)}\Big(\big(2(\lambda^\natural + \rho_0|\alpha^\natural) + 2m(k+h^\vee) - n(\alpha|\alpha)\big)^2 \\ &+ 2(\lambda^\natural|\lambda^\natural + 2\rho_0) - (k+1)^2\Big) + \frac{h^\vee - 2}{8}, \qquad \alpha \in \Delta_{1/2},\end{aligned} \tag{6.12}$$

$$\begin{aligned}\mathcal{N}^{\,\theta}_{n,m}(k, \Lambda, h) = h &- \frac{1}{4(k+h^\vee)}\Big(\big(m(k+h^\vee) - n\big)^2 + 2(\lambda^\natural|\lambda^\natural + 2\rho_0) \\ &- (k+1)^2\Big) + \frac{h^\vee - 2}{8}.\end{aligned} \tag{6.13}$$

$\lambda^\natural$ in these formulas stands for

$$\lambda^\natural = \Lambda + \rho^\natural_{1/2}. \tag{6.14}$$

The partition function $P_W(\eta)$ is defined with respect to the root system Δ_W of the Ramond sector of the vertex algebra $W_k(\mathfrak{g}, x)$. The set of positive roots Δ_W^+ is a disjoint union of

$$\begin{gathered}\{(\alpha, m)|\alpha \in \Delta_0, m \in \mathbb{N}\}, \{(\alpha, 0)|\alpha \in \Delta_0^+\}, \\ \{(\alpha^\natural, m)|\alpha \in \Delta_{1/2}, m \in \mathbb{N}\}, \{(\alpha^\natural, 0)|\alpha \in \Delta_{1/2}^+\}, \\ \{(0, m)|m \in \mathbb{N}\},\end{gathered} \tag{6.15}$$

where the multiplicity of the last set is r.

6.3 General twisted sector

Let $\widehat{\eta} = (\eta, s), \eta \in \mathfrak{h}^{\natural *}, s \in \mathbb{R}$. The determinant $\det^{\mathrm{tw}}_{\widehat{\eta}}(k, \Lambda, h)$ of the Verma module with highest weight (Λ, h) on the weight space $(\Lambda-\eta, h+s)$ is given by

$$\begin{aligned}
\det{}^{\mathrm{tw}}_{\widehat{\eta}}(k,\Lambda,h) &= (k+h^\vee)^{(r-1)\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}_{n,m}^{\ \theta}(k,\Lambda,h)^{P_W(\widehat{\eta}-(0,mn))} \\
&\times \prod_{\substack{\beta\in\Delta_0^+ \\ m\in-\epsilon_\beta+\mathbb{Z}, \\ m>0, n\in\mathbb{N}}} \mathcal{N}_{n,m}^{\ -\beta}(k,\Lambda)^{q(\beta,n)P_W(\widehat{\eta}-n(-\beta,m))} \prod_{\substack{\beta\in\Delta_0^+ \\ m\in\epsilon_\beta+\mathbb{Z}, \\ m\geq 0, n\in\mathbb{N}}} \mathcal{N}_{n,m}^{\ \beta}(k,\Lambda)^{q(\beta,n)P_W(\widehat{\eta}-n(\beta,m))} \\
&\times \prod_{\substack{\alpha\in\Delta_{1/2}\setminus\Delta_{1/2}^+ \\ m\in\epsilon_\alpha+1/2+\mathbb{Z}, \\ m>0, n\in\mathbb{N}}} \mathcal{N}_{n,m}^{\ \alpha}(k,\Lambda,h)^{q(\alpha,n)P_W(\widehat{\eta}-n(\alpha^\natural,m))} \prod_{\substack{\alpha\in\Delta_{1/2}^+ \\ m\in\epsilon_\alpha+1/2+\mathbb{Z}, \\ m\geq 0, n\in\mathbb{N}}} \mathcal{N}_{n,m}^{\ \alpha}(k,\Lambda,h)^{q(\alpha,n)P_W(\widehat{\eta}-n(\alpha^\natural,0))},
\end{aligned} \tag{6.16}$$

where

$$\mathcal{N}_{n,m}^{\ \beta}(k,\Lambda) = (\lambda^\natural+\widetilde{\rho}^\natural|\beta) + m(k+h^\vee) - \tfrac{n}{2}(\beta|\beta), \qquad \beta\in\Delta_0, \tag{6.17}$$

$$\begin{aligned}
\mathcal{N}_{n,m}^{\ \alpha}(k,\Lambda,h) =& h - \frac{1}{4(k+h^\vee)}\Big(\big(2(\lambda^\natural+\widetilde{\rho}^\natural|\alpha^\natural) + 2m(k+h^\vee) - n(\alpha|\alpha)\big)^2 \\
&+ 2(\lambda^\natural|\lambda^\natural+2\widetilde{\rho}^\natural) - \big(k+h^\vee-(\widetilde{\rho}|\theta)\big)^2 \\
&+ 2k\sum_{\gamma\in\Delta_+}(-1)^{p_\gamma}\epsilon_\gamma(1-\epsilon_\gamma)\Big) - h^{\mathrm{ch}} - h^{\mathrm{ne}}, \qquad \alpha\in\Delta_{1/2},
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
\mathcal{N}_{n,m}^{\ \theta}(k,\Lambda,h) =& h - \frac{1}{4(k+h^\vee)}\Big(\big(m(k+h^\vee)-n\big)^2 + 2(\lambda^\natural|\lambda^\natural+2\widetilde{\rho}^\natural) \\
&- \big(k+h^\vee-(\widetilde{\rho}|\theta)\big)^2 + 2k\sum_{\gamma\in\Delta_+}(-1)^{p_\gamma}\epsilon_\gamma(1-\epsilon_\gamma)\Big) - h^{\mathrm{ch}} - h^{\mathrm{ne}}.
\end{aligned} \tag{6.19}$$

The twistings are assumed to be in the range $0\leq\epsilon_\alpha<1$ for all $\alpha\in\Delta_+$, the numbers h^{ch} and h^{ne} are given in (5.21) and (5.22) respectively, $\lambda^\natural$ in these formulas stands for

$$\lambda^\natural = \Lambda + \frac{1}{2}\sum_{\alpha\in\Delta_{1/2}}(-1)^{p_\alpha}\alpha(\epsilon_\alpha+\varkappa_\alpha), \tag{6.20}$$

where $\varkappa_\alpha$ is defined in (5.26).

The function $P_W(\eta)$ is the partition function of the set of positive roots Δ_W^+ of the W-algebra $W_k(\mathfrak{g},x)$ in the twisted sector. Δ_W^+ is a disjoint union of

$$\begin{gathered}
\{(\alpha,m)|\alpha\in\Delta_0^+, m\in\epsilon_\alpha+\mathbb{Z}, m\geq 0\}, \{(-\alpha,m)|\alpha\in\Delta_0^+, m\in-\epsilon_\alpha+\mathbb{Z}, m>0\}, \\
\{(\alpha^\natural,m)|\alpha\in\Delta_{1/2}^+, m\in\epsilon_\alpha+1/2+\mathbb{Z}, m\geq 0\}, \\
\{(\alpha^\natural,m)|\alpha\in\Delta_+\setminus\Delta_{1/2}^+, m\in\epsilon_\alpha+1/2+\mathbb{Z}, m>0\}, \{(0,m)|m\in\mathbb{N}\},
\end{gathered} \tag{6.21}$$

where again the multiplicity of the last set is r.

6.4 Derivation of the determinant formula

We prove here the determinant formula for minimal W-algebras, stated in sections 6.1, 6.2, 6.3. To avoid very long expressions in the derivation we will do part of the calculations for two most important cases: the NS (untwisted) sector (5.28) and the Ramond sector (5.29).

The determinant formula in the untwisted case was obtained in [23], we include its derivation here for completeness. The NS and Ramond sectors are special cases of the general twisting (see section 4.2). The computations in the general twisted case (as anywhere in this paper the case $\epsilon(\mathfrak{h}) = 0$ only is discussed) are similar to the those presented in this section.

The factors of the W-algebra $W(\mathfrak{g}, x)$ determinant formula are generically the same as the factors of the underlying twisted loop algebra $\widetilde{\mathfrak{g}}$ determinant formula, the factors just have to be reexpressed in terms of the W-algebra weights. The determinant formula vanishes if and only if there is a singular vector in the Verma module. We recall from section 4.4 that if there is a singular vector $\big|\, \widehat{s} \,\big\rangle$ in the highest weight module $R_{\widehat{\lambda}}$ of the affine vertex algebra $V_k(\mathfrak{g})$ then there is a singular vector $\big|\, s \,\big\rangle_k$ in the corresponding highest weight module $Q_k(\lambda)$ of the vertex algebra $\mathcal{C}(\mathfrak{g}, x, k) = V_k(\mathfrak{g}) \otimes F(A_{\rm ch}) \otimes F(A_{\rm ne})$ and it is given by $\big|\, s \,\big\rangle_k = \big|\, \widehat{s} \,\big\rangle \times \big|\, 0 \,\big\rangle_{\rm ch} \times \big|\, 0 \,\big\rangle_{\rm ne}$. The vector $\big|\, s \,\big\rangle_k$ is d_0-closed and if in addition it is not d_0-exact then it is a singular vector in the W-algebra module $M_k(\lambda) = H_0(Q_k(\lambda), d_0)$. We will see that generically the vector $\big|\, s \,\big\rangle_k$ is not d_0-exact apart from a few cases listed in the Corollary below. It also comes out that all the W-algebra singular vectors are given by the above construction. It is proved by the standard degree counting, which shows that there is no room for other factors in the determinant formula.

First we reexpress the factors of the affine determinant formula (3.22) in terms of the W-algebra weights. Substituting $\widehat{\rho} = (\widetilde{\rho}, h^\vee, 0)$ into (3.22) one gets the factors

$$\varphi^{\,\alpha}_{n,m}(\lambda, k) = (\lambda + \widetilde{\rho}|\alpha) + m(k + h^\vee) - \tfrac{n}{2}(\alpha|\alpha), \tag{6.22}$$

where $\widehat{\alpha} = (\alpha, 0, m)$ is a positive root of the twisted loop algebra $\widetilde{\mathfrak{g}}$. The "twisted rho" $\widetilde{\rho}$ is defined in (3.16). In the NS and Ramond cases it is equal to

$$\text{NS: } \widetilde{\rho} = \rho = \rho_0 + \tfrac{1}{2}(h^\vee - 1)\theta, \tag{6.23}$$

$$\text{Ramond: } \widetilde{\rho} = \rho_0 + \tfrac{1}{2}\theta. \tag{6.24}$$

In the untwisted case $\widetilde{\rho} = \rho$ in agreement with a well known fact that the Weyl vector of affine Lie superalgebra is equal to $\widehat{\rho} = (\rho, h^\vee, 0)$. We will show now that the Weyl vector $\widehat{\rho} = (\widetilde{\rho}, h^\vee, 0)$ of a twisted loop algebra in the Ramond sector satisfies the set of equations (3.15), proving the conjecture of section 3.1 for the special case of Ramond twisting. The simple roots of $\widetilde{\mathfrak{g}}$ are

$$\begin{aligned} \widehat{\alpha}_s &= (\alpha_s, 0, 0), \\ \widehat{\beta}_s &= (\beta_s - \theta, 0, 1/2), \\ \widehat{\theta} &= (\theta, 0, 0), \end{aligned} \tag{6.25}$$

where α_s and β_s are simple roots of $\mathfrak{g}$, such that $(\alpha_s|\theta) = 0$ and $(\beta_s|\theta) = 1/2$. Compute the products of $\widehat{\rho} = (\widetilde{\rho}, h^\vee, 0)$, where $\widetilde{\rho} = \rho_0 + \frac{1}{2}\theta$, with our simple roots: $(\widehat{\rho}|\widehat{\alpha}_s) = (\rho_0|\alpha_s) = \frac{1}{2}(\alpha_s|\alpha_s)$ since ρ_0 is the "rho" vector for the set of roots Δ_0^+; $(\widehat{\rho}|\widehat{\beta}_s) = (\widetilde{\rho}|\beta_s) - 1 + \frac{h^\vee}{2} = (\rho|\beta_s) = \frac{1}{2}(\beta_s|\beta_s) = \frac{1}{2}(\widehat{\beta}_s|\widehat{\beta}_s)$ since $\rho = \widetilde{\rho} + \frac{1}{2}(h^\vee - 2)\theta$; and $(\widehat{\rho}|\widehat{\theta}) = 1$ is trivial.

The factor $\varphi^{\,\alpha}_{n,m}(\lambda, k)$ vanishes if and only if there is a singular vector in the $\widetilde{\mathfrak{g}}$ module which appears first time on the weight space $\widehat{\lambda} - \widehat{\eta}$, where $\widehat{\eta} = (n\alpha, 0, nm)$. Then (with exception of the cases listed in the Lemma below) there is also a singular vector in the $W_k(\mathfrak{g},x)$ module with weights $(\Lambda(\lambda,k),h(\lambda,k))$ appearing first time on the weight space $\left(\Lambda-n\alpha^\natural,h+n\left(m+\frac{1}{2}(\alpha|\theta)\right)\right)$.

Let us first discuss the case $(\alpha|\theta)=0$, which happens when $\alpha\in\Delta_0$ or $\alpha=0$. Then the factor is expressed as

$$\varphi_{n,m}^{\ \alpha}(\lambda,k)=(\lambda^\natural+\widetilde{\rho}^\natural|\alpha)+m(k+h^\vee)-\tfrac{n}{2}(\alpha|\alpha),\quad \alpha(x)=0. \tag{6.26}$$

Substituting $\lambda^\natural$ by a shifted Λ using (5.25) one gets exactly the factor entering to the W-algebra determinant formula. The factor doesn't depend on h. The corresponding singular vector appears on the weight space $(\Lambda-n\alpha,h+nm)$.

Now we proceed to the case $(\alpha|\theta)\neq 0$. We would like to collect the factors which give rise to the W-algebra module singular vectors on the weight space $(\Lambda-n\alpha^\natural,h+nm)$. These are two factors $\varphi_{n,m-\frac{1}{2}(\alpha|\theta)}^{\ \alpha}$ and $\varphi_{n,m+\frac{1}{2}(\alpha|\theta)}^{\ \bar{\alpha}}$, where $\bar{\alpha}$ is a "mirror" of the root α:

$$\bar{\alpha}=\alpha^\natural-\tfrac{1}{2}(\alpha|\theta)\theta. \tag{6.27}$$

So the following expression is the W-algebra determinant factor for the singular vectors on the weight space $(\Lambda-n\alpha^\natural,h+nm)$:

$$\begin{aligned}\mathcal{N}_{n,m}^{\ \alpha}&\equiv-\frac{1}{(k+h^\vee)(\alpha|\theta)^2}\,\varphi_{n,m-\frac{1}{2}(\alpha|\theta)}^{\ \alpha}\,\varphi_{n,m+\frac{1}{2}(\alpha|\theta)}^{\ \bar{\alpha}}=\\&=\frac{1}{(k+h^\vee)(\alpha|\theta)^2}\left(\tfrac{1}{4}(\alpha|\theta)^2\left((\lambda+\widetilde{\rho}|\theta)-k-h^\vee\right)^2-\right.\\&\left.-\left((\lambda^\natural+\widetilde{\rho}^\natural|\alpha^\natural)+m(k+h^\vee)-\tfrac{n}{2}(\alpha|\alpha)\right)^2\right),\quad\alpha\in\Delta_>.\end{aligned} \tag{6.28}$$

Next one has to express $\mathcal{N}_{n,m}^{\ \alpha}$ in terms of Λ and h. For that reason we rewrite (5.30) and (5.31) as

$$\text{NS: } h=\frac{1}{4(k+h^\vee)}\left(2(\lambda^\natural|\lambda^\natural+2\rho_0)+((\lambda|\theta)-k-1)^2-(k+1)^2\right), \tag{6.29}$$

$$\text{Ramond: }\begin{aligned}h=&\frac{1}{4(k+h^\vee)}\left(2(\lambda^\natural|\lambda^\natural+2\rho_0)+\left((\lambda|\theta)-k-h^\vee+1\right)^2-\right.\\&\left.-(k+1)^2\right)-\frac{h^\vee-2}{8},\end{aligned} \tag{6.30}$$

where in the Ramond sector formula we used the fact that $\operatorname{sdim}\mathfrak{g}_{1/2}=2h^\vee-4$ in the case of minimal gradation. Note that the part including the $(\lambda|\theta)$ term is the same as in (6.28). So one can rewrite the determinant factor as

$$\text{NS: }\begin{aligned}\mathcal{N}_{n,m}^{\ \alpha}=h-&\frac{1}{4(k+h^\vee)}\left(2(\lambda^\natural|\lambda^\natural+2\rho_0)-(k+1)^2\right.\\&\left.+\frac{4}{(\alpha|\theta)^2}\left((\lambda^\natural+\rho_0|\alpha^\natural)+m(k+h^\vee)-\tfrac{n}{2}(\alpha|\alpha)\right)^2\right),\end{aligned} \tag{6.31}$$

$$\text{Ramond: }\begin{aligned}\mathcal{N}_{n,m}^{\ \alpha}=h-&\frac{1}{4(k+h^\vee)}\left(2(\lambda^\natural|\lambda^\natural+2\rho_0)-(k+1)^2\right.\\&\left.+\frac{4}{(\alpha|\theta)^2}\left((\lambda^\natural+\rho_0|\alpha^\natural)+m(k+h^\vee)-\tfrac{n}{2}(\alpha|\alpha)\right)^2\right)+\frac{h^\vee-2}{8},\end{aligned} \tag{6.32}$$

where $\alpha \in \Delta_>$, $\lambda^\natural = \Lambda$ in the NS case and $\lambda^\natural = \Lambda + \rho^\natural_{1/2}$ in the Ramond case.

Now we should check which of the singular vectors of type $|\,\widehat{s}\,\rangle \times |\,0\,\rangle_{\rm ch} \times |\,0\,\rangle_{\rm ne}$ ($|\,\widehat{s}\,\rangle$ is a singular vector in the affine vertex algebra highest weight module $R_{\widehat{\lambda}}$) are d_0-exact, and so trivial in the d_0-homology.

Let $|\,\lambda\,\rangle_k = |\,\widehat{\lambda}\,\rangle \times |\,0\,\rangle_{\rm ch} \times |\,0\,\rangle_{\rm ne}$ be a highest weight vector in the highest weight module $Q_k(\lambda)$ of the vertex algebra $\mathcal{C}(\mathfrak{g},x,k) = V_k(\mathfrak{g}) \otimes F(A_{\rm ch}) \otimes F(A_{\rm ne})$. Define the space ξ as a span of vectors of the form $u_{\theta,-n_1} u_{\theta,-n_2} \cdots u_{\theta,-n_l} |\,\lambda\,\rangle_k$, $n_i \in \mathbb{N}$.

Lemma. *In the module $Q_k(\lambda)$ all the vectors of the form $|\,\widehat{t}\,\rangle \times |\,0\,\rangle_{\rm ch} \times |\,0\,\rangle_{\rm ne}$, where $|\,\widehat{t}\,\rangle$ is a vector in the affine vertex algebra module $R_{\widehat{\lambda}}$, which are d_0-exact or differ from the highest weight vector $|\,\lambda\,\rangle_k$ by a d_0-exact vector, belong to the space ξ or to the space $u_{\alpha,-1+\epsilon_\alpha}\,\xi$, where $\alpha \in \Delta_{1/2}$, $1/2 < \epsilon_\alpha < 1$ or $\alpha \in \Delta_{1/2} \backslash \Delta^-_{1/2}$, $\epsilon_\alpha = 1/2$.*

In the case of minimal gradation d_0 has the following form:

$$\begin{aligned} d_0 = & \sum_{\substack{\alpha\in\Delta_{1/2} \\ n\in\epsilon_\alpha+1/2+\mathbb{Z}}} (-1)^{p_\alpha} u_{\alpha,n-1/2} b^\alpha_{-n} + \sum_{n\in\mathbb{Z}} u_{\theta,n-1} b^\theta_{-n} \\ & - \frac{1}{2} \sum_{\substack{\alpha\in\Delta^+_{1/2} \\ n_1\in\epsilon_\alpha+1/2+\mathbb{Z} \\ n_2\in-\epsilon_\alpha+1/2+\mathbb{Z}}} \mathrm{f}^\theta_{\theta-\alpha,\alpha}\, c_{\gamma,n_1+n_2} b^{\theta-\alpha}_{-n_2} b^\alpha_{-n_1} + (f|u_\theta) b^\theta_0 + \sum_{\substack{\alpha\in\Delta_{1/2} \\ n\in\epsilon_\alpha+1/2+\mathbb{Z}}} \psi_{\alpha,n} b^\alpha_{-n}. \end{aligned} \tag{6.33}$$

One has to analyze the action of d_0 on the ghost number 1 vector $c^\alpha_n |\,\eta\,\rangle$, $|\,\eta\,\rangle \in \xi$. The action of d_0 on other type ghost number 1 vectors obviously can not give a vector of the form $|\,\widehat{t}\,\rangle \times |\,0\,\rangle_{\rm ch} \times |\,0\,\rangle_{\rm ne}$. One can easily check that

$$d_0\, c_{\theta,n} |\,\eta\,\rangle = (r_1\, u_{\theta,n-1} + r_2\, \delta_{0,n})\, |\,\eta\,\rangle, \qquad n \in -\mathbb{N}_0 \tag{6.34}$$

(r_1 and r_2 are nonzero constants), proving that any vector in ξ is in the same homology class as zero or the highest weight vector $|\,\lambda\,\rangle_k$. In the same way

$$\begin{aligned} d_0\, c_{\alpha,m} |\,\eta\,\rangle = \Big(r_3\, u_{\alpha,m-1/2} + r_4\, \psi_{\alpha,m} + r_5 \sum_{m_1\in-\epsilon_\alpha+1/2+\mathbb{Z}} c_{\gamma,m+m_1} b^{\theta-\alpha}_{-m_1} \Big) |\,\eta\,\rangle, \\ \alpha \in \Delta_{1/2},\ m < 1/2,\ m \in \epsilon_\alpha + 1/2 + \mathbb{Z}, \end{aligned} \tag{6.35}$$

where r_3, r_4, r_5 are nonzero constants. Now we observe that the second and the third terms in (6.35) vanish or proportional to $|\,\eta\,\rangle$ if and only if $0 < m < 1/2$ or $m = 0$, $\alpha \in \Delta_{1/2} \backslash \Delta^-_{1/2}$, and by this complete the proof of the lemma.

Corollary. *The singular vectors of the form $|\,\widehat{s}\,\rangle \times |\,0\,\rangle_{\rm ch} \times |\,0\,\rangle_{\rm ne}$ which do not give rise to the singular vectors in the W-algebra highest weight module are $(u_{\theta,-1})^n |\,\lambda\,\rangle_k$ and $(u_{\alpha,-1+\epsilon_\alpha})^n |\,\lambda\,\rangle_k$, $\alpha \in \Delta_{1/2} \backslash \Delta^-_{1/2}$, $\epsilon_\alpha = 1/2$, $n \in \mathbb{N}$.*

The direct analysis of the structure of the twisted loop algebra module singular vectors shows that these are the only singular vectors which belong to the spaces ξ and $u_{\alpha,-1+\epsilon_\alpha}\,\xi$ discussed in the Lemma above.

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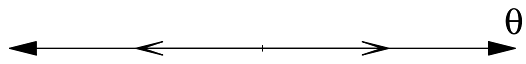

Figure 1: $osp(1|2)$ root system.

Next we just collect all the factors eliminating those corresponding to the singular vectors which belong to the spaces listed in the Lemma. This elimination affects the lower bound for the running indices in the determinant formula. And at the end we get the factors of the determinant formulas in sections 6.1, 6.2, 6.3.

The degrees of the factors should not be less than the W-algebra partitions since the singular vector generates a maximal submodule in the W-algebra highest weight module. So one gets that the expressions in (6.5), (6.10), (6.16) are divisors of the determinant formulas. Now by using the usual counting arguments coming from the estimation of the power of the determinant formula one shows that actually the powers in these expressions are already equal to the maximal estimation and therefore there is no room for other factors. We will not reproduce here the degree counting, since it is fully equivalent to the one performed in [23] for the untwisted case.

7. Examples

In this section we discuss all the simple Lie superalgebras of rank up to 2 except $sl(2)$: $sl(3), so(5), G_2, osp(1|2), sl(2|1), osp(3|2), osp(1|4), psl(2|2)$. The $sl(2)$ Lie algebra is not presented here since there is only untwisted quantum reduction on it. The $osp(1|2)$ superalgebra is of rank 1, all the rest in the list above are of rank 2.

In all the examples the quantum reduction corresponding to the minimal gradation on the Lie superalgebra is discussed. The minimal gradation is generated by the $sl(2)$ embedding associated to the highest root θ as it is described in section 2.

The generating element x is normalized as $(x|x) = 1/2$ (since $x \equiv \theta/2$ and $(\theta|\theta) = 2$). In the case of rank 2 algebra the orthogonal to x Cartan generator is denoted by y: $y \in \mathfrak{h}, (y|x) = (y|\theta) = 0$. We normalize it by $(y|y) = \pm 1/2$, where the sign should be chosen according to the sign of the metric in the corresponding root space direction: $(y|y) = -1/2$ for $sl(2|1), osp(3|2), psl(2|2)$ Lie superalgebras and $(y|y) = 1/2$ for the rest of the rank-2 examples.

We discuss only the case when the Cartan subalgebra is untwisted: $\epsilon(x) = \epsilon(y) = 0$. One should note that in all the rank-2 examples there is yet another possibility: $\epsilon(y) = 1/2$, which is not studied in the current paper.

At the beginning of each example there is a figure showing the root system of the Lie superalgebra under discussion. Even and odd roots are shown by arrows of different style.

7.1 $osp(1|2)$

The $osp(1|2) \approx B(0,1)$ algebra is the simplest Lie superalgebra. It is of rank 1. The root system is shown on figure 1. There is only one good gradation on it: the obvious Dynkin gradation. The quantum reduction procedure on $osp(1|2)$ is described in [22] and [23]: one

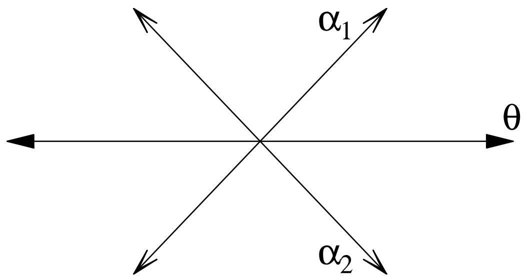

Figure 2: $sl(2|1)$ root system.

gets the famous $N = 1$ superconformal algebra, an extension of the Virasoro algebra by a dimension-3/2 fermionic primary field.

The $osp(1|2)$ algebra has two positive roots: θ and $\theta/2$. The dual Coxeter number is $h^\vee = \frac{1}{2}\,\mathrm{sdim}\,\mathfrak{g}_{1/2} + 2 = 3/2$. The central charge is given by (5.2):

$$c = \frac{k}{k+3/2} - 6k - 5/2\,. \tag{7.1}$$

There are two possible twistings: $\epsilon \equiv \epsilon_{\theta/2} = 0$ leads to half-integer modes of the dimension 3/2 operator (the NS sector); $\epsilon = 1/2$ leads to integer modes (the Ramond sector).

With a help of formula (6.3) one gets the well known [20, 17, 26] determinant formula for the $N = 1$ superconformal algebra:

$$\det{}_\eta(k,h) = \prod_{\substack{m,n\in\mathbb{N},\\ m-n\in 2\mathbb{Z}+2\epsilon}} \nu_{n,m}(k,h)^{P_W(\eta - \frac{mn}{2})}, \tag{7.2}$$

where

$$\nu_{n,m}(k,h) = h - \frac{1}{4(k+\frac{3}{2})}\left(\left(m(k+\frac{3}{2}) - \frac{n}{2}\right)^2 - (k+1)^2\right) - \frac{\epsilon}{8}\,. \tag{7.3}$$

The partition function of the $N = 1$ superconformal algebra can be easily expressed using the following generating function:

$$\prod_{l=1}^{\infty} \frac{1 + x^{l-1/2+\epsilon}}{1 - x^l} = \sum_{n\in(\frac{1}{2}+\epsilon)\mathbb{N}_0} P_W(n)x^n. \tag{7.4}$$

7.2 $sl(2|1)$

The $sl(2|1) \approx A(1,0)$ algebra is a rank-2 Lie superalgebra with 4 even and 4 odd generators. The root system (see figure 2) consists of one pair of even roots $(\theta, -\theta)$ and two pairs of odd isotropic roots $(\alpha_1, -\alpha_1; \alpha_2, -\alpha_2)$. The system of positive roots, for which θ is a highest root, is $\Delta_+ = \{\alpha_1, \alpha_2, \theta = \alpha_1 + \alpha_2\}$. The product between the simple roots is $(\alpha_1|\alpha_2) = 1$. The dual Coxeter number is $h^\vee = 1$.

There is one Dynkin gradation (it is also the minimal one), which corresponds to $f = u_{-\theta}$. All other good gradations may be obtained from the Dynkin one by changing the element x, which generates the gradation. With respect to the minimal gradation $\Delta_0 = \varnothing, \Delta_{1/2} = \{\alpha_1, \alpha_2\}, \Delta_{-1/2} = \{-\alpha_1, -\alpha_2\}$.

The W-algebra obtained from the minimal gradation is the $N = 2$ superconformal algebra. It is generated by four fields: the Virasoro field $L \sim J^{\{f\}}$, two dimension-3/2 fermionic fields $G^+ \sim J^{\{-\alpha_2\}}$ and $G^- \sim J^{\{-\alpha_1\}}$, and one dimension-1 bosonic current $J = 2J^{\{y\}}$, where y is defined in the beginning of section 7. One has to introduce "2" in the definition of the U(1) current J to get it conventionally normalized: then G^+ and G^- have U(1) charges +1 and −1 respectively. The explicit expressions for the fields can be found in [22], section 7. The central charge is

$$c = -6k - 3. \tag{7.5}$$

If $\epsilon(\mathfrak{h}) = 0$ then there is a one parameter family of twistings: $\epsilon_{\alpha_2} \equiv \epsilon$, $\epsilon_{\alpha_1} = 1 - \epsilon$. The modes of the bosonic fields L and J are integer, $n \in -\epsilon + 1/2 + \mathbb{Z}$ in G^+_n and $m \in \epsilon + 1/2 + \mathbb{Z}$ in G^-_m. The untwisted case $\epsilon = 0$ leads to the NS sector, the case $\epsilon = 1/2$ gives the Ramond sector. Different sectors (different ϵ) are isomorphic to the NS sector, the isomorphism is given by the so called U(1) flow [29].

The separation of $\Delta_{1/2}$ and $\Delta_{-1/2}$ to positive and negative parts (see section 5.2) is made with respect to $y \in \mathfrak{h}^\natural$: $\Delta^+_{1/2} = \{\alpha_1\}$, $\Delta^+_{-1/2} = \{-\alpha_2\}$, $\Delta^-_{1/2} = \{\alpha_2\}$, $\Delta^-_{-1/2} = \{-\alpha_1\}$. So in the Ramond sector G^+_0 annihilates the highest weight state (while G^-_0 does not), since G^+ is associated to the root $-\alpha_2 \in \Delta^+_{-1/2}$. The "rho" vectors are $\rho_0 = 0$ and $\rho_{1/2} = -\frac{1}{2}\alpha_1$.

Although the determinant formula for the general twisting can be obtained from the NS sector determinant formula using the U(1) flow, we want to use our general formulae from sections 6.2, 6.3. For the NS and Ramond sectors the determinant formula is

$$\begin{aligned} \det_{\widehat{\eta}}(k,q,h) &= (k+1)^{\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,\theta}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-(0,mn))} \\ &\times \prod_{m\in\frac{1}{2}-\epsilon+\mathbb{N}_0} \mathcal{N}^{\,\alpha_1}_{1,m}(k,q,h)^{P^{(1,m)}_W(\widehat{\eta}-(1,m))} \prod_{m\in\frac{1}{2}+\epsilon+\mathbb{N}_0} \mathcal{N}^{\,\alpha_2}_{1,m}(k,q,h)^{P^{(-1,m)}_W(\widehat{\eta}-(-1,m))}, \end{aligned} \tag{7.6}$$

where

$$\mathcal{N}^{\,\theta}_{n,m}(k,q,h) = h - \frac{1}{4(k+1)}\Big((m(k+1)-n)^2 - (q-\epsilon)^2 - (k+1)^2\Big) - \frac{\epsilon^2}{2}, \tag{7.7}$$

$$\mathcal{N}^{\,\alpha_1(\alpha_2)}_{1,m}(k,q,h) = h - m\big(m(k+1) \mp (q-\epsilon)\big) - \frac{\epsilon^2}{2} + \frac{k+1}{4}, \tag{7.8}$$

$q = 2\Lambda$ is the J_0 eigenvalue, $\epsilon = 0(1/2)$ in the NS (Ramond) sector. This determinant formula was obtained in [4] and in [25].

Applying the formulae in section 6.3 to the $sl(2|1)$ Lie superalgebra one gets the determinant formula of the $N = 2$ algebra for the case of general twisting ϵ. It is not surprising that after a simplification the determinant formula becomes the same as in the case of NS and Ramond sectors (7.6)–(7.8), the only modification is that now ϵ is a continuous parameter in the range $-1/2 < \epsilon \leq 1/2$.

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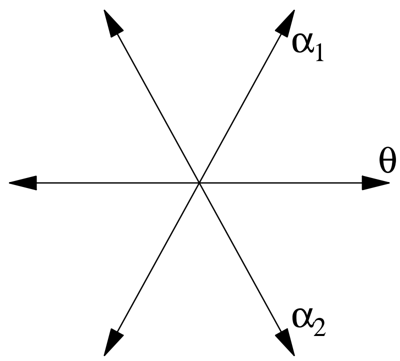

Figure 3: $sl(3)$ root system.

The partitions are expressed as coefficients of the power expansion of the following generating functions:

$$\prod_{l=1}^{\infty} \frac{(1+x\,y^{l-1/2-\epsilon})(1+x^{-1}y^{l-1/2+\epsilon})}{(1-y^l)^2} = \sum P_W(n_1,n_2)\,x^{n_1}y^{n_2}, \tag{7.9}$$

$$\frac{1}{1+x^{j_1}y^{j_2}} \prod_{l=1}^{\infty} \frac{(1+x\,y^{l-1/2-\epsilon})(1+x^{-1}y^{l-1/2+\epsilon})}{(1-y^l)^2} = \sum P_W^{(j_1,j_2)}(n_1,n_2)\,x^{n_1}y^{n_2}. \tag{7.10}$$

The determinant formula for the case when the orthogonal Cartan generator $y \in \mathfrak{h}^{\natural}$ is twisted ($\epsilon(y) = 1/2$) is calculated using quantum reduction in [24].

7.3 $sl(3)$

The root system of the $sl(3) \approx A_2$ Lie algebra is shown on figure 3. We use the normalization $(\alpha|\alpha) = 2$, the product of simple roots is $(\alpha_1|\alpha_2) = -1$, the dual Coxeter number is $h^\vee = 3$. The quantum reduction on $sl(3)$ is very similar to the quantum reduction on $sl(2|1)$, the difference comes from the fact that all the roots of $sl(3)$ are even. The quantum reduction of the minimal gradation on $sl(3)$ is described in [2], it leads to the Bershadsky–Polyakov algebra. The algebra resembles the $N = 2$ superconformal algebra, but the dimension-3/2 generators G^+ and G^- are bosonic, and their operator product expansion contains non-linear terms of type $:JJ:$. The central charge of the algebra is

$$c = \frac{8\,k}{k+3} - 6k - 1\,. \tag{7.11}$$

All the discussion of the previous subsection is applicable here. There also exists a U(1) flow. The determinant formula is

$$\begin{aligned} \det_{\widehat{\eta}}(k,q,h) &= (k+3)^{\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,\theta}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-(0,mn))} \\ &\times \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}-\epsilon+\mathbb{N}_0}} \mathcal{N}^{\,\alpha_1}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(1,m))} \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}+\epsilon+\mathbb{N}_0}} \mathcal{N}^{\,\alpha_2}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(-1,m))}, \end{aligned} \tag{7.12}$$

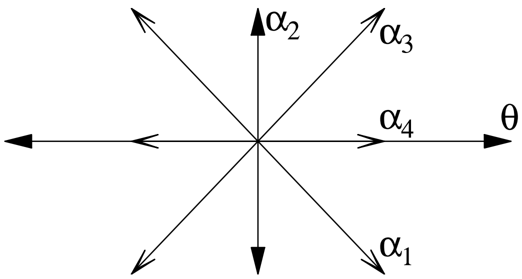

Figure 4: $osp(3|2)$ root system.

where

$$\mathcal{N}^{\ \theta}_{n,m}(k,q,h) = h - \frac{1}{4(k+3)}\Big((m(k+3)-n)^2 + 3(q+\epsilon)^2 - (k+1)^2\Big) + \frac{\epsilon^2}{2}, \quad (7.13)$$

$$\mathcal{N}^{\alpha_1(\alpha_2)}_{n,m}(k,q,h) = h - \frac{1}{4(k+3)}\Big(\big(2m(k+3) - 2n \pm 3(q+\epsilon)\big)^2 + + 3(q+\epsilon)^2 - (k+1)^2\Big) + \frac{\epsilon^2}{2}, \quad (7.14)$$

$q = \frac{2}{\sqrt{3}}\Lambda$ is the J_0 eigenvalue, $\epsilon \equiv \epsilon_{\alpha_2}$ is taken in the range $-1/2 < \epsilon \leq 1/2$, in particular $\epsilon = 0$ corresponds to the NS sector, $\epsilon = 1/2$ – to the Ramond sector.

The partition generating function is

$$\prod_{l=1}^{\infty} \frac{1}{(1 - x\, y^{l-1/2-\epsilon})(1 - x^{-1} y^{l-1/2+\epsilon})(1-y^l)^2} = \sum P_W(n_1,n_2)\, x^{n_1} y^{n_2}. \quad (7.15)$$

7.4 $osp(3|2)$

The $osp(3|2) \approx B(1,1)$ Lie superalgebra has 6 even and 6 odd generators. The root system is shown on figure 4. There are 2 pairs of even roots $\theta, -\theta; \alpha_2, -\alpha_2$ and 3 pairs of odd roots $\alpha_1, -\alpha_1; \alpha_3, -\alpha_3; \alpha_4, -\alpha_4$. The products between simple roots are $(\alpha_1|\alpha_1) = 0$, $(\alpha_2|\alpha_2) = -1/2$, $(\alpha_1|\alpha_2) = 1/2$. With respect to the minimal gradation $\Delta_0 = \{\alpha_2, -\alpha_2\}$, $\Delta_{1/2} = \{\alpha_3, \alpha_4, \alpha_1\}$.

The quantum hamiltonian reduction on the minimal gradation of $osp(3|2)$ gives the $so(3)$ invariant superconformal algebra of [27] (and references therein). Besides the energy–momentum field, there are three bosonic dimension-1 fields generating an affine vertex algebra $V_{-4(k+1/2)}(sl(2))$ and three fermionic dimension-3/2 fields in the triplet representation of the $sl(2)$:

$$\begin{aligned} J^+ &\sim J^{\{\alpha_2\}}, & G^+ &\sim J^{\{-\alpha_1\}}, \\ J &= J^{\{2y\}}, & G &\sim J^{\{-\alpha_4\}}, \\ J^- &\sim J^{\{-\alpha_2\}}, & G^- &\sim J^{\{-\alpha_3\}}. \end{aligned} \quad (7.16)$$

The explicit reduction formulas can be found in [23], section 8.5. For operator product expansions see [27]. The central charge of the W-algebra is

$$c = -6\,k - 7/2. \quad (7.17)$$

The twist numbers are parameterized by one discrete parameter $\sigma \equiv \epsilon_{\alpha_4} = 0$ or $1/2$ and one continuous parameter $\epsilon \equiv \epsilon_{\alpha_1}$. Then $\epsilon_{\alpha_3} = 1 - \epsilon$ and $\epsilon_{\alpha_2} = 1 - \epsilon - \sigma$. The NS sector is obtained if $\epsilon = \sigma = 0$, the Ramond sector is given by $\epsilon = \sigma = 1/2$. There is an isomorphism between different twisted sectors of the algebra (an analogue of the U(1) flow in the case of $N = 2$ superconformal algebra):

$$\begin{array}{ll} J^+_n \mapsto J^+_{n-\epsilon}\,, & G^+_n \mapsto G^+_{n-\epsilon}\,, \\ J^-_n \mapsto J^-_{n+\epsilon}\,, & G^-_n \mapsto G^-_{n+\epsilon}\,, \end{array} \quad \begin{array}{l} J_n \mapsto J_n + \epsilon(2k+1)\delta_{n,0}\,, \\ L_n \mapsto L_n - \epsilon J_n - \epsilon^2(k+\tfrac{1}{2})\delta_{n,0}\,, \\ G_n \mapsto G_n\,. \end{array} \tag{7.18}$$

The general twisted sector of the W-algebra with $\sigma = 0$ ($\sigma = 1/2$) is isomorphic to the NS sector (Ramond sector). We state below only the NS and Ramond sectors determinant formulas, the determinant formula for the general twisted sector may be obtained from the determinant formula for the NS or Ramond sector using this isomorphism.

Inserting the quantities $h^\vee = 1/2$, $\rho_0 = -1/2\alpha_2$, $\rho_{1/2} = -1/2\alpha_3$ into the W-algebra determinant formula one gets the following expression

$$\begin{aligned} \det_{\widehat{\eta}}(k,q,h) = {} & (k+\tfrac{1}{2})^{\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{\substack{m,n\in\mathbb{N},\\ m-n\in 2\mathbb{Z}+2\sigma}} \nu_{n,m}(k,h)^{P_W(\widehat{\eta}-(0,\frac{mn}{2}))} \\ & \times \prod_{m\in\frac{1}{2}-\sigma+\mathbb{N}_0} \mathcal{N}^{\alpha_3}_{1,m}(k,q,h)^{P_W^{(1,m)}(\widehat{\eta}-(1,m))} \prod_{m\in\frac{1}{2}+\sigma+\mathbb{N}_0} \mathcal{N}^{\alpha_1}_{1,m}(k,q,h)^{P_W^{(-1,m)}(\widehat{\eta}-(-1,m))} \\ & \times \prod_{n\in\mathbb{N},\, m\in\mathbb{N}_0} \mathcal{N}^{\alpha_2}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(1,m))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{-\alpha_2}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(-1,m))}, \end{aligned} \tag{7.19}$$

where

$$\begin{aligned} \nu_{n,m}(k,q,h) = {} & h + \frac{k}{4} + \frac{3(1-\sigma)}{8} - \\ & - \frac{1}{4(k+\frac{1}{2})}\Big(\big(m(k+\frac{1}{2}) - \frac{n}{2}\big)^2 - (q + \frac{1}{2} - \sigma)^2\Big), \end{aligned} \tag{7.20}$$

$$\mathcal{N}^{\alpha_1(\alpha_3)}_{1,m}(k,q,h) = h - m^2(k+\frac{1}{2}) \mp m(q + \frac{1}{2} - \sigma) + \frac{k}{4} + \frac{3(1-\sigma)}{8}\,, \tag{7.21}$$

$$\mathcal{N}^{\pm\alpha_2}_{n,m}(k,q) = \mp\frac{q+\frac{1}{2}-\sigma}{2} + m(k+\frac{1}{2}) + \frac{n}{4}\,, \tag{7.22}$$

where $q = 2\Lambda$ is the eigenvalue of J_0, and $\epsilon = \sigma = 0$ or $1/2$. The partitions are given by

$$\prod_{l=1}^{\infty} \frac{(1+x\,y^{l-1/2-\sigma})(1+x^{-1}y^{l-1/2+\sigma})(1+y^{l-1/2+\sigma})}{(1-y^l)^2(1-xy^{l-1})(1-x^{-1}y^l)} = \sum P_W(n_1,n_2)\,x^{n_1}y^{n_2}, \tag{7.23}$$

$$\begin{aligned} \frac{1}{1+x^{j_1}y^{j_2}} \prod_{l=1}^{\infty} \frac{(1+x\,y^{l-1/2-\sigma})(1+x^{-1}y^{l-1/2+\sigma})(1+y^{l-1/2+\sigma})}{(1-y^l)^2(1-xy^{l-1})(1-x^{-1}y^l)} & = \\ = \sum P_W^{(j_1,j_2)}(n_1,n_2)\,x^{n_1}y^{n_2}. & \end{aligned} \tag{7.24}$$

The determinant formula was first obtained in [27].

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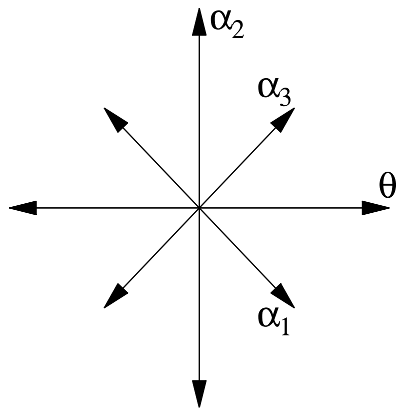

Figure 5: $so(5)$ root system.

7.5 $so(5)$

The root system of $so(5) \approx B_2$ is shown on figure 5. There are two pairs of long roots and two pairs of short roots. The simple roots are α_1 and α_2 with product between them $(\alpha_1|\alpha_2) = -1$. Other positive roots are $\alpha_3 = \alpha_1 + \alpha_2$ and $\theta = 2\alpha_1 + \alpha_2$. With respect to the minimal gradation associated to θ one has $\Delta_0 = \{\alpha_2, -\alpha_2\}, \Delta_{1/2} = \{\alpha_3, \alpha_1\}$.

The quantum reduction corresponding to the minimal gradation on $so(5)$ gives a W-algebra which is generated by the Virasoro field, three bosonic dimension-1 fields forming the $sl(2)$ affine vertex algebra on the level $k_0 = k + 1/2$ ($V_{k+1/2}(sl(2))$), and two bosonic dimension–3/2 fields in the doublet representation of the $sl(2)$:

$$\begin{aligned} J^+ &\sim J^{\{\alpha_2\}}, \\ J &= J^{\{y\}}, \\ J^- &\sim J^{\{-\alpha_2\}}, \end{aligned} \qquad \begin{aligned} G^+ &\sim J^{\{-\alpha_1\}}, \\ G^- &\sim J^{\{-\alpha_3\}}. \end{aligned} \tag{7.25}$$

The central charge of the W-algebra is

$$c = \frac{10\,k}{k+3} - 6k - 1. \tag{7.26}$$

The twistings are parameterized by one continuous parameter $\epsilon \equiv \epsilon_{\alpha_1}$. Other twist numbers are related to ϵ as $\epsilon_{\alpha_3} = -\epsilon$, $\epsilon_{\alpha_2} = -2\epsilon$. The NS sector corresponds to $\epsilon = 0$, the Ramond sector is obtained when $\epsilon = 1/2$. The isomorphism connecting different sectors is given by

$$\begin{aligned} &J_n^+ \mapsto J_{n-2\epsilon}^+\,, \quad G_n^+ \mapsto G_{n-\epsilon}^+\,, \quad J_n \mapsto J_n - \epsilon(k+\tfrac{1}{2})\delta_{n,0}\,, \\ &J_n^- \mapsto J_{n+2\epsilon}^-\,, \quad G_n^- \mapsto G_{n+\epsilon}^-\,, \quad L_n \mapsto L_n - 2\epsilon J_n + \epsilon^2(k+\tfrac{1}{2})\delta_{n,0}\,. \end{aligned} \tag{7.27}$$

To write down the determinant formula for the W-algebra one uses $h^\vee = 3$, $\rho_0 = \alpha_2/2$,

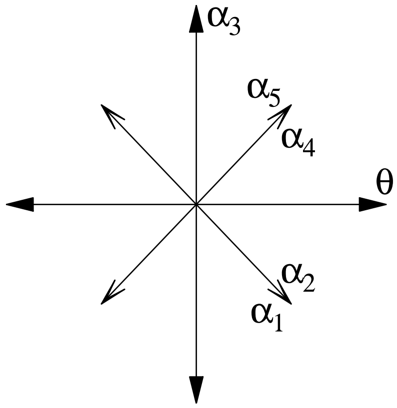

Figure 6: $psl(2|2)$ root system.

$\rho^{\natural}_{1/2} = \alpha^{\natural}_3/2$. The determinant formula for the NS and Ramond sector is

$$\begin{aligned}\det_{\widehat{\eta}}(k,q,h) &= (k+3)^{\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\ \theta}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-(0,mn))} \\ &\times \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}+\epsilon+\mathbb{N}_0}} \mathcal{N}^{\ \alpha_1}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(-1/2,m))} \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}-\epsilon+\mathbb{N}_0}} \mathcal{N}^{\ \alpha_3}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(1/2,m))} \\ &\times \prod_{n\in\mathbb{N},\, m\in\mathbb{N}_0} \mathcal{N}^{\ \alpha_2}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(1,m))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\ -\alpha_2}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(-1,m))},\end{aligned} \tag{7.28}$$

where

$$\mathcal{N}^{\ \theta}_{n,m}(k,q,h) = h - \frac{1}{4(k+3)}\Big(\big(m(k+3)-n\big)^2 + \\ + (2q+1+\epsilon)^2 - k^2 - 2k - 2\Big) + \frac{\epsilon^2}{2}, \tag{7.29}$$

$$\mathcal{N}^{\alpha_1(\alpha_3)}_{n,m}(k,q,h) = h - \frac{1}{4(k+3)}\Big(\big(\mp(2q+1+\epsilon)+2m(k+3)-n\big)^2 + \\ + (2q+1+\epsilon)^2 - k^2 - 2k - 2\Big) + \frac{\epsilon^2}{2}, \tag{7.30}$$

$$\mathcal{N}^{\pm\alpha_2}_{n,m}(k,q) = \pm(2q+1+\epsilon) + m(k+3) - n\,, \tag{7.31}$$

here $q = \Lambda$ is the J_0 eigenvalue and $\epsilon = 0$ or $1/2$. The partition function is

$$\prod_{l=1}^{\infty} \frac{1}{(1-x^{1/2}y^{l-1/2-\epsilon})(1-y^l)^2(1-x^{-1/2}y^{l-1/2+\epsilon})} \\ \times \frac{1}{(1-x\,y^{l-1})(1-x^{-1}\,y^l)} = \sum P_W(n_1,n_2)\,x^{n_1}y^{n_2}. \tag{7.32}$$

7.6 $psl(2|2)$

The root system of the $psl(2|2) = sl(2|2)/\mathbb{C}I \approx A(1,1)$ Lie superalgebra is shown on figure 6. The metric in the α_3 direction of the root space is negative. The even part of the algebra is just $sl(2)\oplus sl(2)$, leading to the 2 pairs of even roots: $\theta,-\theta;\alpha_3,-\alpha_3$. There are 8 odd roots: $\alpha_1,\alpha_2,\alpha_4,\alpha_5$ and their opposites, all of them are isotropic. The $A(n,n)$ type Lie superalgebras have an interesting feature: the number of simple roots is greater than the rank of the algebra, in our case the number of simple roots is 3, and the rank is 2. The roots α_1 and α_2 (as well as α_4 and α_5) coincide since the action of Cartan generators on the corresponding root elements of the Lie superalgebra is identical, so the odd root spaces are two-dimensional. To resolve the "degeneracy" of the odd roots one should equip the root system with an additive "charge": the "charge" of the even roots being zero, the "charge" of α_1 and α_4 being 1 and the "charge" of α_2 and α_5 being -1.

The set of simple roots for which θ is a highest root is $\{\alpha_1,\alpha_2,\alpha_3\}$. The products between simple roots are $(\alpha_1|\alpha_1)=(\alpha_2|\alpha_2)=0$, $(\alpha_3|\alpha_3)=-2$, $(\alpha_3|\alpha_1)=(\alpha_3|\alpha_2)=1$. Other positive roots are obtained from the simple ones as $\alpha_4=\alpha_1+\alpha_3$, $\alpha_5=\alpha_2+\alpha_3$, $\theta=\alpha_1+\alpha_5=\alpha_2+\alpha_4$.

The W-algebra obtained by the quantum reduction of the minimal gradation on $psl(2|2)$ is the $N=4$ superconformal algebra [1]. This is a Lie algebra generated by the Virasoro field L, three bosonic dimension-1 fields J, J^+ and J^-, forming an $sl(2)$ affine vertex algebra on level $k_0=-k-1$, and four fermionic dimension–3/2 fields $G^+,G^-,\bar{G}^+,\bar{G}^-$ in two doublet representations of the $sl(2)$:

$$\begin{array}{c} J^+\sim J^{\{\alpha_3\}},\\ J=J^{\{y\}},\\ J^-\sim J^{\{-\alpha_3\}}, \end{array}\qquad \begin{array}{cc} G^+\sim J^{\{-\alpha_1\}}, & \bar{G}^+\sim J^{\{-\alpha_2\}},\\ G^-\sim J^{\{-\alpha_4\}}, & \bar{G}^-\sim J^{\{-\alpha_5\}}. \end{array} \tag{7.33}$$

The central charge of the algebra is

$$c=-6(k+1). \tag{7.34}$$

The operator product expansions of the algebra and the explicit reduction formulas can be found in [23] (section 8.4).

The twistings are parameterized by two numbers: $\epsilon_1\equiv\epsilon_{\alpha_1}$ and $\epsilon_2\equiv\epsilon_{\alpha_2}$. Other twistings are expressed as $\epsilon_{\alpha_3}=-\epsilon_1-\epsilon_2$, $\epsilon_{\alpha_4}=-\epsilon_2$, $\epsilon_{\alpha_5}=-\epsilon_1$. The NS sector is given by $\epsilon_1=\epsilon_2=0$, the Ramond sector corresponds to $\epsilon_1=\epsilon_2=1/2$. There is a U(1) flow, which relates different sectors [29], in particular NS and Ramond sectors are isomorphic.

The dual Coxeter number of $psl(2|2)$ is $h^\vee=0$. In our case $\Delta_0^+=\{\alpha_3\}$, $\Delta_{1/2}^+=\{\alpha_4,\alpha_5\}$, therefore $\rho_0=\alpha_3/2$ and $\rho_{1/2}^\natural=-\alpha_4^\natural$. Denoting by $q=\Lambda$ the J_0 eigenvalue we write down the determinant formula of the $N=4$ superconformal algebra in the case of NS ($\epsilon=\epsilon_1=\epsilon_2=0$) and Ramond ($\epsilon=\epsilon_1=\epsilon_2=1/2$) sectors:

$$\begin{aligned} \det{}_{\widehat{\eta}}(k,q,h) &= k^{\sum_{m,n\in\mathbb{N}}P_W(\widehat{\eta}-(0,mn))}\prod_{m,n\in\mathbb{N}}\mathcal{N}^{\,\theta}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-(0,mn))}\\ &\times\prod_{m\in\frac{1}{2}+\epsilon+\mathbb{N}_0}\mathcal{N}^{\,\alpha_1}_{1,m}(k,q,h)^{2P_W^{(-1/2,m)}(\widehat{\eta}-(-1/2,m))}\prod_{m\in\frac{1}{2}-\epsilon+\mathbb{N}_0}\mathcal{N}^{\,\alpha_4}_{1,m}(k,q,h)^{2P_W^{(1/2,m)}(\widehat{\eta}-(1/2,m))}\\ &\times\prod_{n\in\mathbb{N},\,m\in\mathbb{N}_0}\mathcal{N}^{\,\alpha_3}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(1,m))}\prod_{m,n\in\mathbb{N}}\mathcal{N}^{\,-\alpha_3}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(-1,m))}, \end{aligned} \tag{7.35}$$

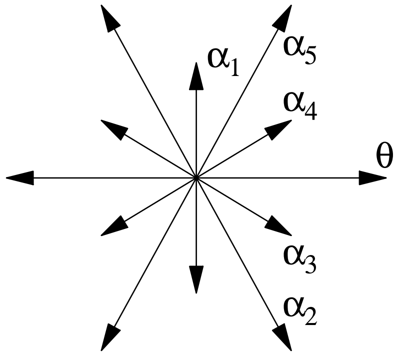

Figure 7: G_2 root system.

where

$$\mathcal{N}_{n,m}^{\ \theta}(k,q,h) = h - \frac{1}{4k}\Big((km-n)^2 - 4(q+1/2-\epsilon)^2\Big) + \frac{k+2}{4} - \epsilon^2, \tag{7.36}$$

$$\mathcal{N}_{1,m}^{\alpha_1(\alpha_4)}(k,q,h) = h - m\big(km \pm 2(q+1/2-\epsilon)\big) + \frac{k+2}{4} - \epsilon^2, \tag{7.37}$$

$$\mathcal{N}_{n,m}^{\pm\alpha_3}(k,q) = \mp 2(q+1/2-\epsilon) + km + n. \tag{7.38}$$

The partition generating functions are

$$\prod_{l=1}^{\infty} \frac{(1+x^{1/2}y^{l-1/2-\epsilon})^2(1+x^{-1/2}y^{l-1/2+\epsilon})^2}{(1-y^l)^2(1-xy^{l-1})(1-x^{-1}y^l)} = \sum P_W(n_1,n_2)\, x^{n_1}y^{n_2}, \tag{7.39}$$

$$\frac{1}{1+x^{j_1}y^{j_2}} \prod_{l=1}^{\infty} \frac{(1+x^{1/2}y^{l-1/2-\epsilon})^2(1+x^{-1/2}y^{l-1/2+\epsilon})^2}{(1-y^l)^2(1-xy^{l-1})(1-x^{-1}y^l)} = \\ = \sum P_W^{(j_1,j_2)}(n_1,n_2)\, x^{n_1}y^{n_2}. \tag{7.40}$$

The determinant formulae were conjectured in [18].

7.7 G_2

This is a simple exceptional Lie algebra with 14 generators. The root system is shown on figure 7. G_2 is the only simple Lie algebra with a root square ratio equal to 3. There are 6 long and 6 short roots. The simple roots are α_1 and α_2, the products between them are $(\alpha_1|\alpha_1) = 2/3, (\alpha_2|\alpha_2) = 2, (\alpha_1|\alpha_2) = -1$. Other positive roots are $\alpha_3 = \alpha_2 + \alpha_1, \alpha_4 = \alpha_2 + 2\alpha_1, \alpha_5 = \alpha_2 + 3\alpha_1, \theta = 2\alpha_2 + 3\alpha_1$.

The W-algebra obtained by the quantum reduction procedure from the minimal gradation on G_2 is generated by the Virasoro field, three dimension-1 bosonic fields forming the $sl(2)$ affine vertex algebra on level $k_0 = 3k+5$ and four bosonic dimension–3/2 fields in the quadruplet of the $sl(2)$:

$$\begin{aligned} J^+ &\sim J^{\{\alpha_1\}}, \\ J &= J^{\{\sqrt{3}y\}}, \\ J^- &\sim J^{\{-\alpha_1\}}, \end{aligned} \qquad \begin{aligned} G^{++} &\sim J^{\{-\alpha_2\}}, \\ G^{+} &\sim J^{\{-\alpha_3\}}, \\ G^{-} &\sim J^{\{-\alpha_4\}}, \\ G^{--} &\sim J^{\{-\alpha_5\}}. \end{aligned} \tag{7.41}$$

The central charge of the algebra is

$$c = \frac{14k}{k+4} - 6k\,. \tag{7.42}$$

The twist numbers are parameterized by one discrete parameter σ, which can take values 0 and 1/2, and one continuous parameter ϵ: $\epsilon_{\alpha_1} = 2\epsilon, \epsilon_{\alpha_2} = -3\epsilon + \sigma, \epsilon_{\alpha_3} = -\epsilon + \sigma, \epsilon_{\alpha_4} = \epsilon + \sigma, \epsilon_{\alpha_5} = 3\epsilon + \sigma$. One gets the NS sector when $\epsilon = \sigma = 0$, the Ramond sector is obtained when $\epsilon = 0, \sigma = 1/2$.

The dual Coxeter number of G_2 is $h^\vee = 4$, The Weyl vectors of $\Delta_0 = \{\alpha_1, -\alpha_1\}$ and $\Delta_{1/2} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ are $\rho_0 = \alpha_1/2$ and $\rho_{1/2} = (\alpha_4 + \alpha_5)/2$. The determinant formula of the minimal $W_k(G_2)$ algebra in the NS and Ramond sectors becomes

$$\begin{aligned}
\det{}_{\widehat{\eta}}(k,q,h) &= (k+4)^{\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,\theta}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-(0,mn))} \\
&\times \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}+\sigma+\mathbb{N}_0}} \mathcal{N}^{\,\alpha_2}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(-3/2,m))} \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}-\sigma+\mathbb{N}_0}} \mathcal{N}^{\,\alpha_5}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(3/2,m))} \\
&\times \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}+\sigma+\mathbb{N}_0}} \mathcal{N}^{\,\alpha_3}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(-1/2,m))} \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}-\sigma+\mathbb{N}_0}} \mathcal{N}^{\,\alpha_4}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(1/2,m))} \\
&\times \prod_{n\in\mathbb{N},\, m\in\mathbb{N}_0} \mathcal{N}^{\,\alpha_1}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(1,m))} \prod_{m,n\in\mathbb{N}} \mathcal{N}^{\,-\alpha_1}_{n,m}(k,q)^{P_W(\widehat{\eta}-n(-1,m))},
\end{aligned} \tag{7.43}$$

where

$$\begin{aligned}
\mathcal{N}^{\,\theta}_{n,m}(k,q,h) = h - \frac{1}{4(k+4)}\Big(&\big(m(k+4)-n\big)^2 + \\
&+ \frac{4}{3}(q+2\sigma)(q+2\sigma+1) - (k+1)^2\Big) + \sigma^2,
\end{aligned} \tag{7.44}$$

$$\begin{aligned}
\mathcal{N}^{\alpha_2(\alpha_5)}_{n,m}(k,q,h) = h - \frac{1}{4(k+4)}\Big(&4\big(m(k+4) - n \mp (q+2\sigma+1/2)\big)^2 + \\
&+ \frac{4}{3}(q+2\sigma)(q+2\sigma+1) - (k+1)^2\Big) + \sigma^2,
\end{aligned} \tag{7.45}$$

$$\begin{aligned}
\mathcal{N}^{\alpha_3(\alpha_4)}_{n,m}(k,q,h) = h - \frac{1}{4(k+4)}\Big(&4\big(m(k+4) - \frac{1}{3}n \mp \frac{1}{3}(q+2\sigma+1/2)\big)^2 + \\
&+ \frac{4}{3}(q+2\sigma)(q+2\sigma+1) - (k+1)^2\Big) + \sigma^2,
\end{aligned} \tag{7.46}$$

$$\mathcal{N}^{\pm\alpha_1}_{n,m}(k,q) = \pm 2/3(q+2\sigma+1/2) + m(k+4) - n/3\,, \tag{7.47}$$

here $q = \sqrt{3}\Lambda$ is the J_0 eigenvalue and $\sigma = 0$ or $1/2$. The partition function is

$$\begin{aligned}
&\prod_{l=1}^{\infty} \frac{1}{(1-x^{3/2}y^{l-1/2-\sigma})(1-x^{1/2}y^{l-1/2-\sigma})(1-x^{-1/2}y^{l-1/2+\sigma})(1-x^{-3/2}y^{l-1/2+\sigma})} \\
&\qquad\qquad \times \frac{1}{(1-x\,y^{l-1})(1-y^l)^2(1-x^{-1}\,y^l)} = \sum P_W(n_1,n_2)\, x^{n_1} y^{n_2}.
\end{aligned} \tag{7.48}$$

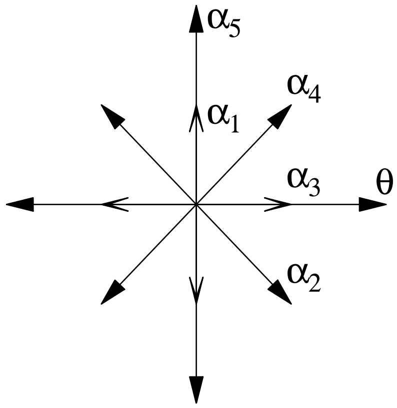

Figure 8: $osp(1|4)$ root system.

7.8 $osp(1|4)$

The $osp(1|4) \approx B(0,2)$ Lie superalgebra has 14 generators: 10 even and 4 odd. The even subalgebra is the $so(5)$ Lie algebra. The root system is shown on figure 8. There are 8 even roots and 4 odd roots. The metric in both directions of the root space is positive. The simple roots are α_1 and α_2. The defining products are $(\alpha_1|\alpha_1) = 1/2$, $(\alpha_2|\alpha_2) = 1$ and $(\alpha_1|\alpha_2) = -1/2$. Other positive roots are given by $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = 2\alpha_1 + \alpha_2$, $\alpha_5 = 2\alpha_1$, $\theta = 2\alpha_3$.

The minimal W-algebra obtained from the quantum reduction of $osp(1|4)$ is generated along with the Virasoro field by five dimension-1 fields which form an $osp(1|2)$ affine vertex algebra on level $k_0 = k + 1$ and three dimension–3/2 fields in the triplet representation of the $osp(1|2)$:

$$\begin{aligned} J^+ &\sim J^{\{\alpha_5\}}, & & \\ j^+ &\sim J^{\{\alpha_1\}}, & G^+ &\sim J^{\{-\alpha_2\}}, \\ J &= J^{\{y\}}, & G &\sim J^{\{-\alpha_3\}}, \\ j^- &\sim J^{\{-\alpha_1\}}, & G^- &\sim J^{\{-\alpha_4\}}, \\ J^- &\sim J^{\{-\alpha_5\}}, & & \end{aligned} \tag{7.49}$$

j^+, j^-, G are fermionic fields, the rest are bosonic. The central charge of the algebra is

$$c = \frac{6k}{k + 5/2} - 6k - 3/2. \tag{7.50}$$

There is a 2 parameter family of twistings: $\epsilon \equiv \epsilon_{\alpha_1}$ is continuous parameter, $\sigma = \epsilon_{\alpha_3}$ is discrete, can take value 0 and 1/2. Other twistings are expressed as $\epsilon_{\alpha_2} = \sigma - \epsilon$, $\epsilon_{\alpha_4} = \sigma + \epsilon$, $\epsilon_{\alpha_5} = 2\epsilon$. The NS sector corresponds to $\epsilon = \sigma = 0$, the Ramond sector is obtained when $\epsilon = 0, \sigma = 1/2$.

Inserting the values of the dual Coxeter number ($h^\vee = 5/2$), of the Weyl vectors for $\Delta_0^+ = \{\alpha_1, \alpha_5\}$ ($\rho_0 = \alpha_5/4$) and for $\Delta_{1/2}^+ = \{\alpha_4\}$ ($\rho_{1/2} = \alpha_4/2$) into our general minimal W-algebra determinant formula in section 6 one gets the determinant formula for the NS and Ramond sectors of the minimal $W_k(osp(1|4))$ algebra:

$$\begin{aligned}
\det_{\widehat{\eta}}(k,q,h) &= (k+\tfrac{5}{2})^{\sum_{m,n\in\mathbb{N}} P_W(\widehat{\eta}-(0,mn))} \prod_{\substack{m,n\in\mathbb{N},\\ m-n\in 2\mathbb{Z}+2\sigma}} \nu_{n,m}(k,h)^{P_W(\widehat{\eta}-(0,\frac{mn}{2}))} \\
&\times \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}-\sigma+\mathbb{N}_0}} \mathcal{N}^{\alpha_4}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(1/2,m))} \prod_{\substack{n\in\mathbb{N},\\ m\in\frac{1}{2}+\sigma+\mathbb{N}_0}} \mathcal{N}^{\alpha_2}_{n,m}(k,q,h)^{P_W(\widehat{\eta}-n(-1/2,m))} \\
&\times \prod_{\substack{n\in\mathbb{N},\, m\in\mathbb{N}_0,\\ m-n\in 2\mathbb{Z}+1}} \nu^{\alpha_5}_{n,m}(k,q)^{P_W(\widehat{\eta}-(\frac{n}{2},\frac{mn}{2}))} \prod_{\substack{m,n\in\mathbb{N},\\ m-n\in 2\mathbb{Z}+1}} \nu^{-\alpha_5}_{n,m}(k,q)^{P_W(\widehat{\eta}-(-\frac{n}{2},\frac{mn}{2}))},
\end{aligned} \tag{7.51}$$

where

$$\nu_{n,m}(k,q,h) = h - \frac{1}{4(k+\frac{5}{2})}\Big(\big(m(k+\frac{5}{2})-\frac{n}{2}\big)^2 + \\ + (2q+\sigma)(2q+\sigma+1)-(k+1)^2\Big)+\frac{\sigma^2}{4}, \tag{7.52}$$

$$\mathcal{N}^{\alpha_4(\alpha_2)}_{n,m}(k,q,h) = h - \frac{1}{4(k+\frac{5}{2})}\Big(\big(2m(k+\frac{5}{2})-n\pm(2q+\sigma+\frac{1}{2})\big)^2 + \\ + (2q+\sigma)(2q+\sigma+1)-(k+1)^2\Big)+\frac{\sigma^2}{4}, \tag{7.53}$$

$$\nu^{\pm\alpha_5}_{n,m}(k,q) = \pm(2q+\sigma+1/2)+m(k+5/2)-n/2\,, \tag{7.54}$$

here $q=\Lambda$ is the eigenvalue of J_0, and $\sigma=0$ for the NS sector and $\sigma=1/2$ for the Ramond sector. The partition generating function is given by

$$\prod_{l=1}^{\infty}\frac{(1+x^{1/2}y^{l-1})(1+y^{l-1/2+\sigma})(1+x^{-1/2}y^l)}{(1-x\,y^{l-1})(1-x^{1/2}y^{l-1/2-\sigma})(1-y^l)^2(1-x^{-1/2}y^{l-1/2+\sigma})(1-x^{-1}y^l)} = \\ = \sum P_W(n_1,n_2)\,x^{n_1}y^{n_2}. \tag{7.55}$$

8. Discussion

We studied the quantum reduction of affine superalgebras in the twisted case. This is also a subject of paper [24]. The methods and the results obtained are essentially the same. However some details and the presentation are different. The main difference is in the choice of the triangular decomposition of the twisted loop algebra. (Compare (3.12) with (2.6–2.9) of [24].) Also the different normal ordered product prescriptions are used (see appendix A). We consider only the case when the Cartan subalgebra is untwisted ($\epsilon(\mathfrak{h})=0$), the discussion in [24] applies to the more general twisting than one discussed here: the case $\epsilon(h)\neq 0$ for some $h\in\mathfrak{h}$ is also allowed.

We would like to show that our main result, the determinant for the Ramond sector of minimal W-algebras is the same as in [24]. Take the determinant formula of Kac and Wakimoto in [24], Theorem 4.2. The Ramond sector corresponds to Example 4.1(b) in [24]. Using the values of s_α from this Example one can evaluate the determinant factors (4.8–4.10) of [24]. Then it is easy to see that the first type factor ($\alpha(x)=0$) coincides with our first type factor (6.11). The other two factors are different by an expression proportional to

$$R = 4(\rho^\natural_{1/2}|\rho^\natural_{1/2} + \rho_0) - \frac{3}{8}\,\sigma - \frac{1}{2}\,h^\vee(h^\vee - 2), \tag{8.1}$$

where $\sigma = 1$, if $\theta/2 \in \Delta$ and $\sigma = 0$ otherwise, $\rho_{1/2}$ and ρ_0 are defined in (5.32) and (5.33) respectively. We prove here that $R = 0$ for any simple Lie superalgebra $\mathfrak{g}$. The proof is based on the fact that the square of the Weyl vector $(\rho|\rho)$ does not depend on the choice of positive roots. We calculate it first for the original choice of positive roots $\Delta_+ = \Delta_0^+ \cup \Delta_{1/2} \cup \{\theta\}$:

$$(\rho|\rho) = (\rho_0|\rho_0) + \frac{1}{2}\left(\frac{1}{2}\,\mathrm{sdim}\mathfrak{g}_{1/2} + 1\right)^2. \tag{8.2}$$

Now we define another set of positive roots $\bar{\Delta}_+$ by "flipping" the roots from $\Delta^-_{1/2}$ to the opposite ones: $\bar{\Delta}_+ = \Delta_0^+ \cup \Delta^+_{1/2} \cup \Delta^+_{-1/2} \cup \{\theta, \sigma\frac{\theta}{2}\}$. This set is "generated" by the element $h_0 + t\,x$, where $h_0 \in \mathfrak{h}^\natural$ is the Cartan element used to split Δ_0 and $\Delta_{1/2}$ to positive and negative parts (see (5.13)), and t is a sufficiently small positive number. Now the new $\bar{\rho}$ is defined with respect to $\bar{\Delta}_+$, and its square

$$(\bar{\rho}|\bar{\rho}) = (\rho_0 + 2\rho^\natural_{1/2}|\rho_0 + 2\rho^\natural_{1/2}) + \frac{1}{2}\left(-\frac{1}{2}\,\sigma + 1\right)^2. \tag{8.3}$$

One can check that $R = (\bar{\rho}|\bar{\rho}) - (\rho|\rho)$. So we proved that $R = 0$ and therefore the determinant factors coincide.

The only factor which is missing in our determinant formula comparing to the formula in [24] is φ_0, which is present only if $\theta/2 \in \Delta$. This factor is a contribution of the $G_0^{\{-\theta/2\}}$ zero mode. But since (unlike [24]) we let this operator act diagonally on the highest weight vector (see (5.16)), we do not have this factor.

The factor multiplicities are given by partition functions defined with respect to Δ^+_W ,the set of positive roots of the minimal W-algebra. The degrees are the same in the present paper and in [24]. (Again up to a small difference in the case when there is a root $\theta/2$: unlike our definition (6.15), in [24] an odd root $(0,0)$ is included in the set of positive W-algebra roots.)

Acknowledgments

The author is grateful to Maria Gorelik for many fruitful discussions and continuous support during the work on this paper. The author also would like to thank Victor Kac for correspondence and Anthony Joseph for comments on the first version of the paper. The results of this paper were presented at the Dublin Institute for Advanced Studies in April 2004 and on the "Algebraic Geometry and Representation Theory Seminar" at the Weizmann institute of Science in May 2004, the author wishes to thank these institutions for the opportunity to give a talk.

A. Normal ordered product conventions

In this appendix we fix the normal ordering conventions. Start from the operator product expansion of two fields:

$$A(z)B(w) = \sum_{l\in N(A,B)-\mathbb{N}_0} \frac{[AB]^{(l)}(w)}{(z-w)^l}, \tag{A.1}$$

where $N(A,B) \in \mathbb{Z}$ is the order of maximal singularity in the operator product expansion of A and B. In all the formulas in this paper the normal ordering sign : : stands for the so called point splitting normal ordering, widely used in a physical literature. It is just the operator product expansion with singular terms removed:

$$:A(z)B(w): = \sum_{l\in\mathbb{N}_0} [AB]^{(-l)}(w)(z-w)^l, \tag{A.2}$$

and then $:AB:(w) = :A(w)B(w):$ is just the zero order term in the operator product expansion of fields $A(z)$ and $B(w)$:

$$:AB: = [AB]^{(0)}. \tag{A.3}$$

In our formalism the normal ordering is affected by local properties of the fields only (when z is close to w). Global properties such as boundary conditions do not influence the normal ordered product.

The normal ordered product is not associative in general, $:(:AB:)C: \neq :A(:BC:):$. However if the fields are free, i.e. the singular part of their mutual operator product expansions include the identity field only (e.g. superghosts and superfermions in this paper), then the normal ordered product is associative.

We introduce mode expansions of the fields

$$\begin{aligned}
A(z) &= \sum_{n\in-\Delta(A)+\epsilon(A)+\mathbb{Z}} A_n\, z^{-n-\Delta(A)}, \\
B(w) &= \sum_{n\in-\Delta(B)+\epsilon(B)+\mathbb{Z}} B_n\, w^{-n-\Delta(B)}, \\
:AB:(w) &= \sum_{n\in-\Delta(A)-\Delta(B)+\epsilon(A)+\epsilon(B)+\mathbb{Z}} :AB:_n\, w^{-n-\Delta(A)-\Delta(B)},
\end{aligned} \tag{A.4}$$

where $\Delta(A), \Delta(B), \Delta(:AB:) = \Delta(A) + \Delta(B)$ are conformal dimensions of correspondent fields. The twistings $\epsilon(A), \epsilon(B) \in \mathbb{R}/\mathbb{Z}$ depend on the boundary conditions. The mode $:AB:_n$ is expressed in terms of A_n and B_m:

$$\begin{aligned}
:AB:_n = &-\sum_{l=1}^{N(A,B)} \binom{\epsilon(A)}{l} [AB]_n^{(l)} + \\
&+ \sum_{m\in-\Delta(A)+\epsilon(A)-\mathbb{N}_0} A_m B_{n-m} + (-1)^{p(A)p(B)} \sum_{m\in-\Delta(A)+1+\epsilon(A)+\mathbb{N}_0} B_{n-m} A_m\,,
\end{aligned} \tag{A.5}$$

$\epsilon(A) \in \mathbb{R}$ can be any number consistent with the algebraic structure of the theory, $p(A)$ and $p(B)$ are the field parities: $p(A) = 0$ if A is even (bosonic) and $p(A) = 1$ if A is odd (fermionic). This formula is derived in appendix E of [28], it also follows from the twisted Borcheds identity. The formula is well known in the untwisted case ($\epsilon(A) = 0$), in the twisted case there are additional terms (the first sum in (A.5)) coming from the singular part of the operator product expansion.

We would like to stress that our definition of the normal ordered product is different from one convenient in the mathematical literature, which uses the separation of a field to "positive" and "negative" parts (see e.g. [19] for details):

$$A(z)_- = \sum_{n \geq -\Delta(A)+1} A_n \, z^{-n-\Delta(A)}, \quad A(z)_+ = \sum_{n < -\Delta(A)+1} A_n \, z^{-n-\Delta(A)}. \tag{A.6}$$

Then in this formalism the normal ordered product ${}^\times_\times A(z)B(w) {}^\times_\times$ is defined as

$${}^\times_\times A(z)B(w) {}^\times_\times = A(z)_+ B(w) + (-1)^{p(A)p(B)} B(w) A(z)_- \, . \tag{A.7}$$

It is easy to show that in the untwisted case the two definitions coincide:

$${}^\times_\times A(z)B(w) {}^\times_\times = :A(z)B(z): \qquad \text{(untwisted case)}. \tag{A.8}$$

But they are in general different in the twisted case.

The advantage of the point-splitting formalism is that expressions in terms of conformal fields (e.g. (3.5), (3.27) or (3.36)) do not change when one changes the boundary conditions.

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